Optimal quantised bits for estimation over a capacity and power limited, lossy channel
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This letter considers the problem of estimating the state of a scalar dynamical system over a wireless channel that is lossy, capacity, and power limited. The limited power assumption, which is valid for most real problems, links the number of quantisation bits and packet error rate. As the number of quantisation bits increases, the quantisation noise decreases but the packet loss rate increases. It is shown that the estimation error variance (EEV) is minimised for a range of quantisation bits. Further, it is shown that operating beyond the optimal range sharply increases the EEV. Simulation results corroborating the analytical results are also presented.

Introduction: Control and estimation of cyber physical systems (CPS) involve wireless communication as it is flexible, cheap, and easy to use. Wireless systems are capacity and power limited and hence lossy. Data transmitted through a wireless system may suffer from quantisation noise and packet loss. So, many researchers have tried to solve problems associated with CPS. These include optimal linear quadratic Gaussian (LOQ) control subject to quantisation noise and packet loss [1], LQG control subject to the packet delay [2], Kalman filtering (KF) in wireless sensor networks [3], and KF with quantised measurements (quantised KF) [4, 5]. The remote KF with 1-bit quantisation has been introduced in [6] and then extended to remote Kalman filtering for multi-bits in [7]. For remote linear estimation, the optimal linear encoder for uniform quantiser was devised in [8], and simultaneous design of the encoder and the quantiser was presented in [9]. KF with transmit power constraint have been addressed in [10–15]. However, these works consider the quantisation noise and packet loss rate as independent.

In this letter, we propose remote estimation of a linear scalar CPS; in a capacity and power-limited, lossy channel. In this model, the encoded data are quantised (b bits/sample), encoded using a modulation and coding scheme (MCS), and transferred from the transmitter to the receiver over a lossy wireless channel. This transfer is typically modelled by an independent Bernoulli packet loss process where the channel is capacity limited [16, 17]. This modelling decouples the quantisation noise and packet error rate and is justified because any degradation in packet loss rate due to increased data rate can be compensated by an increase in power. A capacity-limited and power-limited channel does not offer this luxury, and an increase in data rate due to more quantisation bits increases the packet loss rate (PLR). We show that in such a channel, there is a range of quantised bits in which the EEV is minimised. We characterise this optimal number of quantised bits and study them in various simulation environments. It is observed that the EEV sharply increases when operating outside the optimal range.

System model: We consider a scalar discrete-time linear state dynamical system, given by

\[ x_{k+1} = ax_k + w_k \]

where \( x_k \) is the system state and \( w_k \) is the system state noise at time instant \( k \). The measurement of the system at time \( k \), is given by

\[ y_k = cx_k + v_k \]

where \( y_k \) and \( v_k \) are the system observation and the measurement noise at time instant \( k \). Variables \( a \) and \( c \) are the scalar system and the scalar measurement coefficients, respectively. For a stable system \( |a| < 1 \) and we assume that the measurement is scaled so that \( c = 1 \).

An encoder \( f(\cdot) \) encodes the data to \( u_k \) which is quantised to \( q(u_k) \). This quantised signal \( q(u_k) \) is transmitted using a modulation and coding scheme (MCS) that encodes the data at a rate \( R \), providing a coding gain

**C** and transmits a signal, \( s_k \), from a constellation \( \chi \), \( |\chi| = 2^b \), where \( b = b/R \). The received data suffers from packet loss at time instant \( k \) denoted as \( (\gamma_k) \in [0,1] \), as shown in Figure 1. The extended system model together with the encoder and quantiser at the receiver side can be represented as

\[ x_{k+1} = ax_k + w_k, \quad u_k = f(v_k \cdot 2^{−1}) \cdot z_k = \gamma_k q(u_k) = \gamma_k (u_k + n_k) \]

where the sequences \( Y_k = [\gamma_1, \gamma_2, \ldots, \gamma_k] \) is the quantisation noise, which follows [9, 10, 17] as assumed to be white Gaussian. The probability of packet loss \( \gamma_k \) depends on the MCS scheme chosen and in-particular depends on the signal to noise ratio per bit, \( \rho_{bx} \), and the number of coded bits per sample \( b = b/R \) as shown in Figure 1.

Further, it is shown that operating beyond the optimal range sharply increases the EEV. Simulation results corroborating the analytical results are also presented.
Table 1. Approximate bit error rates for coherent modulations

<table>
<thead>
<tr>
<th>Modulation</th>
<th>$P_b(\infty)$</th>
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<tbody>
<tr>
<td>MPAM:</td>
<td>$P_b \approx \frac{2(M-1)}{M \log_2 M} \left( \frac{690.5 \log_2 M}{M-1} \right)^2$</td>
</tr>
<tr>
<td>MPSK:</td>
<td>$P_b \approx 2 \log_2 M \left( \frac{205 \log_2 M \sin \left( \frac{\pi}{M} \right)}{M-1} \right)^2$</td>
</tr>
<tr>
<td>Rect. MQAM:</td>
<td>$P_b \approx \frac{2(\sqrt{M}-1)}{\sqrt{M} \log_2 M} \left( \frac{330 \log_2 M}{M-1} \right)^2$</td>
</tr>
<tr>
<td>Nonrect. MQAM:</td>
<td>$P_b \approx \frac{4}{\log_2 M} \left( \frac{330 \log_2 M}{M-1} \right)^2$</td>
</tr>
</tbody>
</table>

For the upper bound we let $\epsilon = 0$. Thus letting $\Lambda \rightarrow \infty$, we have

$$p^{CF}(b_\epsilon) = \frac{\sigma_\epsilon^2 + 1 - \epsilon(b_\epsilon) \frac{\sigma_\epsilon^2}{\epsilon} + \frac{\sigma_\epsilon^2}{\epsilon} \mu_\infty}{1 - a^2} \left( \frac{a^2 \epsilon(b_\epsilon) \sigma_\epsilon^2}{\epsilon} + \frac{\sigma_\epsilon^2}{\epsilon} \mu_\infty \right) \left( \frac{a^2 \epsilon(b_\epsilon) \sigma_\epsilon^2}{\epsilon} + \frac{\sigma_\epsilon^2}{\epsilon} \mu_\infty \right)$$

Moreover under the same condition $(1 - a^2)\mu_\infty^2 < \sigma_\epsilon^2$, $EEV_{IL}(b_\epsilon)$ is a decreasing function of $\Lambda$. Thus letting $\Lambda \rightarrow \infty$, we have the universal lower bound as

$$EEV_{IL} = \lim_{b_\epsilon \rightarrow -\infty} EEV_{IL}(b_\epsilon) = \sigma_\epsilon^2 + a^2 \mu_\infty^2$$

For the upper bound we let $\epsilon = 1$. We have

$$EEV_U = \lim_{\epsilon \rightarrow -1} p^{CF}(\epsilon) = p^{CF}(1) = \frac{\sigma_\epsilon^2}{1 - a^2}$$

Figure 2 plots the variations of EEV as $b_\epsilon$ increases for a fixed SNR per bit of 25 dB for square QAM. Observe from the figure that the EEV first decreases and then saturates at a low value, $EEV_{IL}(b_\epsilon)$ as given by Equations (10) and (11) before sharply increasing and then saturating at a higher value, as given by Equation (12). Clearly, there is a range of $b_\epsilon$'s for which EEV is minimised. In what follows we intend to characterise this range. We have

Algorithm 1 Find the optimal $b_\epsilon$'s satisfying Theorem 1.

Require: Inputs $\rho_0, b_\epsilon^2, \sigma_\epsilon^2, \sigma_\epsilon^2$, and Modulation Type (MQAM-MPSK)

1: Initialise $b_\epsilon' \leftarrow 0, b_\epsilon \leftarrow 0, \rho_0 \leftarrow \rho_0$.
2: Given $b_\epsilon$, compute the packet error probabilities $\epsilon(b_\epsilon - 1), \epsilon(b_\epsilon)$, and $\epsilon(b_\epsilon + 1)$ using Equation (8) and Table 1.
3: If both Equations (13) and (14) are satisfied, then $b_\epsilon' \leftarrow b_\epsilon$ and go to Step 2. Else STOP.
3: Else, If $b_\epsilon = b_\epsilon'$, STOP.
4: Else $b_\epsilon \leftarrow b_\epsilon + 1$, and go to Step 2.

Theorem 1. The optimal values of $b_\epsilon$ that minimise the steady-state MSE are the only $b^* \epsilon$ that must satisfy the following simultaneously:

$$-3\epsilon(b_\epsilon - 1)\sigma_\epsilon^2 + C_1 [4\epsilon(b_\epsilon - 1)(A + 1 - \sigma_r)] \geq 0$$

$$-3\epsilon(b_\epsilon - 1)\sigma_\epsilon^2 + C_1 [4\epsilon(b_\epsilon - 1)(A + 1 - \sigma_r)] \geq 0$$

Proof. The solution of the optimisation problem is obtained by solving the following two difference equations.

$$p^{CF}(b_\epsilon + 1) - p^{CF}(b_\epsilon) \geq 0$$

$$p^{CF}(b_\epsilon - 1) - p^{CF}(b_\epsilon) \geq 0$$

Note that $A(b_\epsilon + 1) = 4A(b_\epsilon)$. Employing Equation (9) to evaluate Equations (15) and (16), and after simplification, we have Equations (13) and (14) respectively.

Remark 1. Note that a closed-form expression for the optimal values of $b_\epsilon$ using Equations (13) and (14) is intractable. An iterative algorithm for obtaining the optimal values is presented in Algorithm 1; wherein $\rho_0$ can be calculated from the inputs and $b_\epsilon$ is a preset value to stop the algorithm. Observe that when $EEV_U = EEV_I$, all values are optimal. When $EEV_U \neq EEV_I$, note that $\min_{\rho_0} p(b_\epsilon)$ is greater than or equal to 1 for all the constellations in Table 1. So, $\epsilon(0) = 1$ and $p^{CF}(0) = EEV_U$. Also, as $p^{CF}(\infty) = EEV_I$, it follows that an optimal $b_\epsilon$ will always exist.

Simulations and numerical results: Numerical simulation were performed to evaluate the EEV performance of the proposed system for MQAM and MPSK for both AWGN and Rayleigh Fading Channels for various values of $\sigma_\epsilon^2, \sigma_\epsilon^2, A$, and $N$ for MQAM signal constellation in AWGN.
by observing that the $e(b_i) \to 1$ for the EEV to saturate at EEVU. As in [13], summarising the values in Table 1 as $P_b(b_i) \approx M \mathcal{Q}(\sqrt{\rho_i \beta_M})$ and using (8), we have $\alpha_M \mathcal{Q}(\sqrt{\rho_i \beta_M}) \approx 1 - \frac{1}{\alpha_M}$, where $i \leq 1$ is a positive variable. The threshold on SNR per bit is then given by

$$\rho_i \approx \frac{1}{\beta_M} \left( \frac{1 - \frac{1}{\alpha_M}}{2} \right)^2.$$  \hfill (17)

For MQAM, substituting $i = 0.1, b_i = 1$, and $N = 1000$ in (17), we have $\rho_i^{(t)} = 4.28 \text{ dB.}$ This suggests that more than $4.28 \text{ dB SNR per bit is required for the EEV optimisation to be feasible.}$ This result corresponds to Figure 3a where the curves for $\rho < \rho_i^{(t)}$ are a straight line at EEVU. Likewise with $i = 0.99, \rho_i^{(99)} = 7.9 \text{ dB}$ which roughly gives the SNR per bit when the EEVs reach EEVU. The upper limit of the optimal $b_i$ depends on the SNR per bit. As the SNR per bit increases, the upper limit of optimal $b_i$’s increases. This can be justified by the better BER as the SNR per bit increases so that the packet loss rate saturates for larger $b_i$’s.

Figure 3b plots the variation of EEV with $b_i$ for MPSK modulation. Observe that in addition to the inferences derived from Figures 3a and 2, the optimal range of $b_i$ reduces while the lower limit remains constant. The reduced range is because of larger BER of MPSK as compared to MQAM which results in saturation to MSE for smaller $b_i$’s. The SNR per bit threshold $\rho_i^{(t)}$ for MPSK, with $i = 0.1, b_i = 1$, and $N = 1000$ is $7.23 \text{ dB}$ and $\rho_i^{(99)} = 10.2 \text{ dB}$ which again corresponds well with Figure 3b.

Figure 4 plots the variation of EEV with $b_i$ for MQAM modulation over Rayleigh fading channel. Clearly, the observations made from plots in Figure 3a,b, hold but for higher SNRs. This can be explained by further degradation of average BER performance, $P_b$ as a function of average SNR per bit, $\rho_i$ over Rayleigh fading channels, given by [16] \[ \hat{P}_b(b_i) \approx \frac{\rho_i}{\beta_M} \left( 1 - \frac{1}{\alpha_M} \right)^2. \] There is a corresponding shift in the lower threshold on average SNR per bit which is given by

$$\rho_i \approx \frac{2}{\beta_M} \left( \frac{\alpha_M - 2 \left( 1 - \frac{1}{\alpha_M} \right)^2}{1 - \frac{1}{\alpha_M} - 2 \left( 1 - \frac{1}{\alpha_M} \right)^2} \right).$$  \hfill (18)

For MQAM, with $i = 0.1, b_i = 1$, and $N = 10,000$, $\rho_i^{(t)} = 26.27 \text{ dB}$ and $\rho_i^{(99)} = 49.87 \text{ dB}$ which corresponds well with Figure 4. Observe from all the figures that when the SNR per bit is large enough ($> \rho_i^{(99)}$) then the lower limit of optimal $b_i$’s starts at 3 bits. Further, for such SNR’s the upper limit obtained by using the Chernoff bound is very loose.

**Discussion and conclusions:** We have considered estimation over a lossy, capacity, and power limited channel in this letter. We have shown that there is an optimal range of quantisation bits that minimises the EEV. We have further shown that the upper and lower limits depend on the quantisation noise and BER, respectively. We have shown that these observations are equally valid for Rayleigh fading channels with higher SNRs. The SNR per bit values for which optimisation is feasible have also been provided. The results are similar to other encoder functions, $f(*)$, and have been omitted. For WiFi and LTE, it has been shown that exponential curves model well the packet loss curves in AWGN and inverse SNR curves for Rayleigh channels [10]. As such, the results of this letter will also apply to these systems. Additionally, there is a possibility of various SNR levels in practical systems, which suggests a different adaptive approach to choosing the MCS for optimising the EEV. Future research directions include vector systems, power control, and coding [18].

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