

# On the suitability of the Bandler-Kohout subproduct as an inference mechanism

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**Abstract**—Fuzzy Relational Inference (FRI) systems form an important part of approximate reasoning schemes using fuzzy sets. Especially, the compositional rule of inference (CRI) introduced by Zadeh has attracted the most attention so far. In this work we show that the FRI scheme based on the Bandler-Kohout subproduct along with a suitable realisation of the fuzzy rules, does possess all the important properties cited in favor of using CRI, viz., equivalent and reasonable conditions for their solvability, their interpolative properties and the preservation of the indistinguishability that may be inherent in the input fuzzy sets. Moreover, we show that under certain conditions the equivalence of *First Infer Then Aggregate* and *First Aggregate Then Infer* (FITA and FATI) inference strategies can be shown for the Bandler-Kohout subproduct (BK-Subproduct), much like in the case of CRI. Finally, by way of addressing the computational complexity that may exist in the Bandler-Kohout subproduct we suggest a hierarchical inferencing scheme. Thus this work shows that the BK-subproduct based FRI is as effective and efficient as the CRI itself.

**Keywords**—Fuzzy relational inference systems, Compositional rule of inference, Bandler-Kohout subproduct, fuzzy relational equations, correctness and continuity of inference, hierarchical CRI.

## I. INTRODUCTION

The idea of linguistic fuzzy models imitating the human way of thinking was proposed by L. A. Zadeh in his pioneering work ZADEH [1]. Systems using fuzzy rules and an inference mechanism have been applied in a wide variety of applications, viz., automatic control, decision making, risk analysis, etc.

Let  $X$  be a non-empty classical set. Let us recall that a fuzzy set  $A$  on  $X$  is a mapping from  $X$  to the unit interval, i.e.,  $A : X \rightarrow [0, 1]$ . Let us denote the set of all fuzzy sets on  $X$  by  $\mathcal{F}(X)$ . Given two non-empty classical sets  $X, Y$ , a fuzzy IF-THEN rule is usually given in the following form:

$$\mathbf{IF} \ x \text{ is } A \ \mathbf{THEN} \ y \text{ is } B, \quad (1)$$

where the antecedent fuzzy set  $A \in \mathcal{F}(X)$  and the consequent fuzzy set  $B \in \mathcal{F}(Y)$  represent some properties.

Given a fuzzy observation  $x$  is  $A'$  with  $A' \in \mathcal{F}(X)$ , a corresponding output fuzzy set  $B' \in \mathcal{F}(Y)$ , meaning  $y$  is  $B'$ , is deduced using an *inference mechanism*. Thus, an inference mechanism may be generally viewed as an arbitrary mapping from  $\mathcal{F}(X)$  to  $\mathcal{F}(Y)$  [2], [3].

Many types of inference mechanisms dealing with fuzzy rule based systems have been proposed in the literature and used in practical applications. Of the many fuzzy inference

schemes, fuzzy relational inferences have obtained considerable attention both from theoretical researchers and practitioners. Similarity Based Reasoning (SBR) [4] and Inverse Truth-functional Modification [5] are two of the representative examples of inference mechanisms that do not use fuzzy relations and which are also well established in the literature. However, it should be mentioned that, under certain conditions, an equivalent fuzzy relation based description of some of these inference mechanisms can be given, see [6], [7].

In this work, we will focus only on fuzzy relation based inference mechanism.

### A. Fuzzy Relational Inferences

Fuzzy relational inference mechanisms use a fuzzy relation  $R$  to model a given fuzzy rule base. Here, a fuzzy IF-THEN rule of the form (1) is represented as a fuzzy relation  $R: X \times Y \rightarrow [0, 1]$ , i.e.,  $R \in \mathcal{F}(X \times Y)$ . Then, given a fact  $x$  is  $A'$ , the inferred output  $y$  is  $B'$  is obtained as a composition of  $A'$  and  $R$ , i.e.,

$$B' = A' @ R, \quad (2)$$

where  $A' \in \mathcal{F}(X), B' \in \mathcal{F}(Y)$  and  $@$  is a fuzzy relational composition<sup>1</sup> involving fuzzy logic operations.

Up to now, we discussed only the case of a single fuzzy rule. However, rarely, if ever a single fuzzy rule can be expected to capture the entire knowledge in which the scheme is employed and hence a fuzzy rule base consisting of multiple fuzzy rules is a necessity. Let us consider a fuzzy rule base:

$$\mathbf{IF} \ x \text{ is } A_i \ \mathbf{THEN} \ y \text{ is } B_i, \quad (3)$$

where the fuzzy sets  $A_i \in \mathcal{F}(X)$  and  $B_i \in \mathcal{F}(Y)$ , represent some properties, for  $i = 1, \dots, n$ .

Clearly, for different representations  $R$  of the fuzzy if-then rules in (1) and different compositions  $@$  we obtain different fuzzy relational inference mechanisms, often with varying properties and applicable in specific contexts. We present below two of the most commonly employed fuzzy relations  $R$  to model a given fuzzy rule base and two of the established fuzzy relational inference mechanisms based on different fuzzy compositions.

<sup>1</sup>In literature, we may very often meet a distinction between *fuzzy relational composition* and *image of a fuzzy set under a fuzzy relation*. The first notion denotes a composition of two binary relations while the second one denotes our situation when we compose a fuzzy set and a binary fuzzy relation. However, a unary fuzzy relation (i.e. a fuzzy set) on a universe  $U$  may be viewed as a binary fuzzy relation on a Cartesian product of an empty set and  $U$ . Therefore, we may use the notion *composition* even in our situation and avoid the use of two notions.

### B. Distinct Fuzzy Rule Base Models

- (i) The fuzzy relation  $\check{R} \in \mathcal{F}(X \times Y)$

$$\check{R}(x, y) = \bigvee_{i=1}^n (A_i(x) * B_i(y)), \quad x \in X, y \in Y \quad (4)$$

is the most often used model of fuzzy rules (3) in real world applications. This is mainly due to the successful applications of this, say Cartesian product approach, published by MAMDANI and ASSILIAN in [8], which was followed by a huge number of researchers and practitioners, see e.g. [9], [10].

- (ii) Alternatively, to keep the conditional IF-THEN form of fuzzy rules (3), fuzzy relation  $\hat{R} \in \mathcal{F}(X \times Y)$  given as follows

$$\hat{R}(x, y) = \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)), \quad x \in X, y \in Y \quad (5)$$

can be chosen to model the fuzzy rule base. It deals with a mathematically correct extension of a classical implication denoted by  $\rightarrow$ .

To stress the difference between both the approaches, let us recall the work of DUBOIS et al. [11], where the authors state: "In the view given by (5), each piece of information (fuzzy rule) is viewed as a constraint. This view naturally leads to a conjunctive way of merging the individual pieces of information since the more information, the more constraints and the less possible values to satisfy them." While the same authors describe the second approach proposed by Mamdani and Assilian as follows: "It seems that fuzzy rules modelled by  $\hat{R}$  are not viewed as constraints but are considered as pieces of data. Then the maximum in (4) expresses accumulation of data".

It should be stressed that both approaches have sound logical foundations but from different viewpoints, see e.g. [12], [13], [14]. However, only the approach using  $\check{R}$  was widely used in applications although the implicational approach using  $\hat{R}$  is probably as useful as the Mamdani-Assilian one, see [15]. Nevertheless, as we show in this work, the implicational approach using  $\hat{R}$  does have an important role to play in the case of BK-Subproducts (see Theorem 3.22). For an extensive study of different fuzzy rules we refer to [16], [17], [18].

### C. Coherence

*Consistency* (non-existence of contradictory rules) is a crucial issue to be checked when dealing with a fuzzy rule base. In case of the implicational approach (5) to modelling a fuzzy rule base, the situation gets significantly simpler. It has been noted by DUBOIS et al. [11] that inconsistent rules lower the largest membership degrees in the resulting fuzzy relation. Departing from this fact, they proposed the concept of so-called *coherence* for which an existence of  $y \in Y$  such that  $\hat{R}(x', y) = 1$  for arbitrary  $x' \in X$  is required. This condition is easy to check as well as to ensure, see [11], [19].

An analogous issue in case of the Cartesian product approach  $\check{R}$  has been suggested [20]. However, the condition

has to be redefined and instead of non-emptiness of the core of  $\check{R}$ , its convexity up to some predefined value is required. This approach is unquestionably more complicated and less preferable.

Generally, the consistency (coherence) is a property of the given fuzzy rule base (its model), not the inference mechanism itself. However, as mentioned above and demonstrated in Section IV, each model of a fuzzy rule base is computationally preferable in a combination with a different inference and so, these notions cannot be studied independently.

### D. Compositional Rule of Inference and BK-Subproduct Inference Mechanisms

As noted above, depending on the type of composition @ the fuzzy relational inference varies in its properties. Two of the commonly employed fuzzy relational compositions are the  $\sup - *$  and  $\inf - I$  compositions (see KLIR and YUAN [21]), which when employed lead to the following fuzzy relational inferences.

- (i) The Compositional Rule of Inference (CRI) ZADEH [1] is one of the earliest fuzzy relational inferences. Here, a fuzzy IF-THEN rule of the form (1) is represented as a fuzzy relation  $R(x, y) : X \times Y \rightarrow [0, 1]$ , i.e.,  $R \in \mathcal{F}(X \times Y)$ . Then, given a fact  $x$  is  $A'$ , the inferred output  $B'$  is obtained as composition of  $A'(x)$  and  $R(x, y)$ , i.e.,

$$B'(y) = \bigvee_{x \in X} (A'(x) * R(x, y)), \quad y \in Y \quad (6)$$

where  $*$  is a fuzzy conjunction, typically a t-norm (see KLEMENT et al. [22] for more details). We use the following notation to indicate the CRI scheme:

$$B' = A' \circ R. \quad (7)$$

- (ii) Other than the CRI, let us also recall that it was PEDRYCZ [23] who firstly proposed an inference scheme based on the *Bandler-Kohout subproduct* (BK-Subproduct, for short) proposed by BANDLER and KOHOUT [24], [25], [26].

For a given fuzzy input  $A' \in \mathcal{F}(X)$ , the fuzzy output  $B' \in \mathcal{F}(Y)$  obtained by the BK-Subproduct inference mechanism is defined as follows:

$$B'(y) = \bigwedge_{x \in X} (A'(x) \rightarrow R(x, y)), \quad y \in Y \quad (8)$$

where  $\rightarrow$  is a residual implication (see Sec. II for more details) and  $R$  is the fuzzy relation that the models fuzzy rule (1). We use the following notation to indicate the BK-Subproduct scheme:

$$B' = A' \triangleleft R. \quad (9)$$

We only remark that yet other types of fuzzy relational compositions are studied in literature, for instance, the  $\inf - S$  composition where  $S$  is a t-conorm (see KLIR and YUAN [21]).

### E. A Mathematical Structure for Fuzzy Relational Inference Mechanisms

Based on the above discussion, fuzzy rules may be viewed as a partial mapping from  $\mathcal{F}(X)$  to  $\mathcal{F}(Y)$  assigning  $B_i \in \mathcal{F}(Y)$  to  $A_i \in \mathcal{F}(X)$  for every  $i = 1, \dots, n$ . Then the inference process itself can be viewed as an extension of this partial mapping to a total one [27]. For better understanding, let us adopt the notation from [28] and consider the following structure

$$S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \textcircled{\ast}),$$

where  $X, Y$  are non-empty classical sets,  $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y)$  for all  $i = 1, \dots, n$  are the antecedent and consequent fuzzy sets in the fuzzy rule base,  $\textcircled{\ast} : \mathcal{F}(X) \times \mathcal{F}(X \times Y) \rightarrow \mathcal{F}(Y)$  is a fuzzy relational composition. For instance,  $\textcircled{\ast}$  could be one of  $\circ$  or  $\triangleleft$ . Finally,  $\mathcal{L}$  is an algebra on the unit interval  $[0, 1]$  that provides us with the operations to be employed in the inference process, typically a complete residuated lattice (see Section II-A for more details).

Now, by the choice of the fuzzy relation  $R$  modeling the fuzzy rule base and by the choice of  $\textcircled{\ast}$ , we define a fuzzy function  $f_R^{\textcircled{\ast}}(A) : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$  such that  $f_R^{\textcircled{\ast}}(A) = A \textcircled{\ast} R$ , for an arbitrary  $A \in \mathcal{F}(X)$ .

### F. Studies on the Advantages of CRI and Organization of the Work

Given a fuzzy rule base, the CRI is the most often and widely used fuzzy relational inference for the following reasons:

- (i) An important issue in the applicability of a fuzzy relational inference mechanism  $\textcircled{\ast}$  in a structure  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \textcircled{\ast})$  is to determine an appropriate fuzzy relation  $R$  modelling the given fuzzy rules so as to obtain meaningful conclusions. One of the fundamental properties expected of the corresponding fuzzy adjoint mapping is its *interpolativity*, i.e.,  $f_R^{\textcircled{\ast}}(A_i) = B_i$ . This pertains to the solvability of fuzzy relation equations. In the case of CRI, necessary and sufficient conditions for the solvability of fuzzy relation equations has been well established for long, see e.g. [29]. The state-of-the-art, as well as analogous results, concerning the BK-Subproduct are briefly recalled in Subsection III-A.
- (ii) In PERFILIEVA et al. [28], [30] the authors have dealt with the *continuity* of a fuzzy function  $f_R^{\circ}$  adjoint to the CRI mechanism  $\circ$  in a structure  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \circ)$  and a fuzzy relation  $R$  modelling fuzzy rules (3). The authors give necessary and sufficient conditions for  $f_R^{\circ}$  to be continuous. They have also shown that the concept of continuity is equivalent to the interpolativity of the function  $f_R^{\circ}$ . We follow their ideas in Subsection III-B.
- (iii) KLAWONN and CASTRO [31] have proven two important and interesting results about the CRI scheme and the indistinguishability inherent to the fuzzy sets considered. Firstly, the authors show that the indistinguishability induced by the antecedent fuzzy set of

the rule cannot be overcome. Secondly, they have also demonstrated the *robustness* of fuzzy inference systems employing the CRI mechanism in scenarios where there can be slight discrepancies between the intended input  $A'$  and the actual input  $\widehat{A}$ , i.e.,  $f_R^{\circ}(A') = f_R^{\circ}(\widehat{A})$ . However, this study was done in the case of a single fuzzy rule as in (1). We consider multiple fuzzy rules (3), and prove that in such case this property holds when  $R = \widehat{R}$ . Then we continue with an investigation of this property for the BK-Subproduct. Subsection III-C is devoted to this issue.

- (iv) In the case of CRI, if the input is a fuzzy singleton, then the output determined by the fuzzy adjoint function  $f_R^{\circ}$  depends only on the relation  $R$  modelling the given fuzzy rule base, i.e., the inference plays a role only in case of a non-singleton fuzzy input. A detailed exposition of this topic is done in Subsection IV-A.
- (v) While employing fuzzy relational inferences in a system consisting of multiple fuzzy rules, there are two inference strategies that are usually employed, viz., FATI and FITA. In the case of CRI mechanism, if the fuzzy rules are represented by the fuzzy relation  $\widehat{R}$ , then the FATI inference strategy is equivalent to the FITA inference strategy. However, this is no more true if we employ  $\widehat{R}$  instead of  $\widehat{R}$ . More details on FATI and FITA strategies and their subsequent exploration is done in Section IV-B.
- (vi) Finally, it should be noted that fuzzy relational inferences, in general, are not without their drawbacks due to space and time complexities. Many works have concentrated on increasing the efficiency of the inference process. However, all these have so far been done only for the case of CRI mechanism and especially when the fuzzy rules are represented by the fuzzy relation  $\widehat{R}$ . The above properties will be dealt with in a more detailed way in Subsections IV-C and IV-D.

### G. Motivation for this work: Study of the Bandler-Kohout Subproduct and the relation $\widehat{R}$

From the previous subsection, it is clear that most of the works have tended to concentrate predominantly on the CRI mechanism. However, two facts emerge from it.

On the one hand, the above studies done on CRI can also be conducted for other fuzzy relational inference mechanisms and in this work we intend to do a similar investigation into the Bandler-Kohout Subproduct. On the other hand, note also that some of the advantages available with the CRI mechanism depend to a large extent on the fuzzy rules being modelled by the fuzzy relation  $\widehat{R}$ , which as already noted is appropriate only in the context where the fuzzy rules are viewed as 'positive' pieces of information [11], [17]. However, there are situations when the context dictates to view the fuzzy rules in the conditional nature and the fuzzy relation  $\widehat{R}$  has to be used to model them [15]. Then many of the advantages of the CRI are no more available, viz., the robustness of the CRI mechanism with respect to the indistinguishability of input fuzzy sets in the case of multiple fuzzy rules, the equivalence of FATI and FATI,

the many techniques dealing with enhancing the efficiency of the inference procedures.

In this work, we intend to carry on the following investigation. Firstly, this work shows that all the properties investigated / touted as an advantage of the CRI mechanism is available also for the BK-subproduct and, often, under similar conditions or generality as available on CRI. Secondly, we also highlight that the conditional form of representation of a fuzzy rule base, in conjunction with the BK-subproduct, i.e.,  $f_{\hat{R}}^{\triangleleft}$  is a strong alternative to  $f_{\hat{R}}^{\circ}$  in the appropriate contexts.

## II. MATHEMATICAL BACKGROUND OF FUZZY INFERENCE MECHANISMS

### A. Fuzzy Inference Mechanisms

Fuzzy relational inference mechanisms are mathematically based on a complete residuated lattice (see, e.g., [12]) which we fix for the whole paper as the basic algebraic structure. Let us only briefly recall, that an algebra  $\mathcal{L} = (L, \wedge, \vee, *, \rightarrow, 0, 1)$  is a residuated lattice if

- $(L, \wedge, \vee, 0, 1)$  is a lattice with the least and the greatest element
- $\mathcal{L} = (L, *, 0, 1)$  is a commutative monoid such that  $*$  is isotone in both arguments
- the operation  $\rightarrow$  is a residuation with respect to  $*$ , i.e.

$$a * b \leq c \quad \text{iff} \quad a \rightarrow c \geq b. \quad (10)$$

The following properties [12] are immediately available to us for any  $a, b, c \in \mathcal{L}$ :

$$a = 1 \rightarrow a, \quad (11)$$

$$a \rightarrow c \geq b \rightarrow c \quad \text{whenever} \quad a \leq b, \quad (12)$$

$$a \rightarrow b \leq a \rightarrow c \quad \text{whenever} \quad b \leq c, \quad (13)$$

$$a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c = (b * a) \rightarrow c, \quad (14)$$

$$(a \rightarrow b) \rightarrow b \geq a \vee b, \quad (15)$$

$$a \rightarrow \bigwedge_{i \in \mathcal{I}} b_i = \bigwedge_{i \in \mathcal{I}} (a \rightarrow b_i), \quad (16)$$

$$\bigvee_{i \in \mathcal{I}} (a_i \rightarrow b) \leq \bigwedge_{i \in \mathcal{I}} a_i \rightarrow b, \quad (17)$$

$$\bigvee_{i \in \mathcal{I}} a_i \rightarrow b = \bigwedge_{i \in \mathcal{I}} (a_i \rightarrow b), \quad (18)$$

$$(a \rightarrow b) * c \leq a \rightarrow (b * c). \quad (19)$$

Let us fix the set  $L = [0, 1]$  for the whole paper. Then  $*$  becomes a left-continuous t-norm and  $\rightarrow$  becomes a residual fuzzy implication derived from  $*$ . For more details on these operations we refer to [22], [12], [32].

In  $\mathcal{L}$ , we can derive yet another operation known as the *biresiduum* and defined as follows:

$$a \leftrightarrow b = (a \rightarrow b) \wedge (b \rightarrow a), \quad a, b \in \mathcal{L}. \quad (20)$$

The following properties of the biresiduum will be useful in the sequel (see [12]):

$$(a \leftrightarrow b) * (b \leftrightarrow c) \leq a \leftrightarrow c, \quad (21)$$

$$(a \leftrightarrow b) \wedge (c \leftrightarrow d) \leq (a \wedge c) \leftrightarrow (b \wedge d). \quad (22)$$

Finally, by an extension of an algebraic operation from  $\mathcal{L}$  to operations between fuzzy sets we mean the following:

$$(C \star D)(u, v) = C(u) \star D(v), \quad u \in U, v \in V, \quad (23)$$

where  $\star \in \{\wedge, \vee, *, \rightarrow, \leftrightarrow\}$  and for arbitrary fuzzy sets  $C, D$  on arbitrary universes  $U, V$ , respectively.

### B. Inference Strategies

Now, there are two inference strategies called FITA and FATI, see GOTTWALD [33].

In *First Infer Then Aggregate* (FITA) strategy, we firstly construct individual fuzzy relations  $R_i \in \mathcal{F}(X \times Y)$  from each of the  $n$  fuzzy rules. Then, the given fuzzy observation  $A' \in \mathcal{F}(X)$  is composed with each of these relations  $R_i$  by a chosen inference  $\textcircled{\ast}$  and the obtained individual output fuzzy sets  $B'_i = A' \textcircled{\ast} R_i \in \mathcal{F}(Y)$  are then aggregated to form the final output fuzzy set  $B' \in \mathcal{F}(Y)$ .

In *First Aggregate Then Infer* (FATI) strategy, the individual fuzzy relations  $R_i \in \mathcal{F}(X \times Y)$  from each of the  $n$  fuzzy rules is aggregated into a single fuzzy relation  $R \in \mathcal{F}(X \times Y)$  and this is composed with the given fuzzy observation  $A' \in \mathcal{F}(X)$  to obtain the fuzzy output  $B' = A' \textcircled{\ast} R \in \mathcal{F}(Y)$ .

## III. DESIRABLE PROPERTIES OF INFERENCE MECHANISMS

### A. Interpolativity of Fuzzy Inference Systems - Property (i)

The interpolativity  $f_R^{\textcircled{\ast}}(A_i) = B_i$  is a fundamental property of any inference mechanism. In this case, we say that  $R$  is a *correct model* of given fuzzy rules in the given structure  $S$ . This leads us to deal with a system of fuzzy relation equations [34] where

$$A_i \textcircled{\ast} R = B_i, \quad i = 1, \dots, n, \quad (24)$$

is solved with respect to the known  $A_i \in \mathcal{F}(X), B_i \in \mathcal{F}(Y)$  and unknown  $R \in \mathcal{F}(X \times Y)$ . If  $R$  is a solution to (24) then the adjoint fuzzy function fulfills  $f_R^{\textcircled{\ast}}(A_i) = B_i$ .

In the case of CRI, the above system of equations, reduces to the following:

$$A_i \circ R = B_i, \quad i = 1, \dots, n. \quad (25)$$

Let us recall some main results which may be found, e.g., in [29], [34], [35], [36].

*Theorem 3.1:* System (25) is solvable if and only if  $\hat{R}$  is a solution of the system and moreover,  $\hat{R}$  is the greatest solution of (25).

On the one hand, Theorem 3.1 states the necessary and sufficient condition of the solvability of system (25) and it determines the solution. Moreover, it ensures that the given solution is the greatest one. On the other hand, we still do not know, when  $\hat{R}$  is the solution, i.e., how to ensure the solvability.

*Theorem 3.2 ([36]):* Let  $A_i$  for  $i = 1, \dots, n$  be normal<sup>2</sup>. Then  $\hat{R}$  is a solution of (25) if and only if the following

<sup>2</sup>Let us recall that a fuzzy set  $A$  on a universe  $U$  is called *normal* if there exists an  $x \in U$  such that  $A(x) = 1$ .

condition holds for arbitrary  $i, j \in \{1, \dots, n\}$ :

$$\bigvee_{x \in X} (A_i(x) * A_j(x)) \leq \bigwedge_{y \in Y} (B_i(y) \leftrightarrow B_j(y)). \quad (26)$$

Theorem 3.2 specifies a sufficient condition under which the system is solvable and moreover, it ensures that not only  $\hat{R}$  but also even  $\check{R}$  is a solution of system (25).

It is worth mentioning that condition (26) appearing in Theorem 3.2 is not very convenient in practice. Another sufficient condition for solvability of the systems with a high practical importance was published in [39], [40].

*Theorem 3.3:* Let  $A_i$  for  $i = 1, \dots, n$  be normal and fulfill the Ruspini condition

$$\sum_{i=1}^n A_i(x) = 1, \quad x \in X. \quad (27)$$

Then the system (25) is solvable.

*Remark 3.4:* Besides the case when antecedent fuzzy sets form a fuzzy partition fulfilling the Ruspini condition, the so called  $*$ -semi-partition [37] also plays an important role.

Let us recall what is the state-of-the-art concerning the BK-Subproduct and the interpolativity issue. In the case of BK-Subproduct, the system of equations (24), reduces to the following:

$$A_i \triangleleft R = B_i, \quad i = 1, \dots, n. \quad (28)$$

Concerning system (28), let us recall the following two basic theorems from PEDRYCZ [23] and NOSKOVA [38]:

*Theorem 3.5 ([23]):* System (28) is solvable if and only if  $\check{R}$  is a solution of the system and moreover,  $\check{R}$  is the least solution of system (28).

*Theorem 3.6 ([38], Theorem 2):* Let  $A_i$  for  $i = 1, \dots, n$  be normal. Then  $\hat{R}$  is a solution of (28) if and only if the condition (26) holds for arbitrary  $i, j \in \{1, \dots, n\}$ .

Again, condition (26) to which Theorem 3.6 refers to, is on one hand very transparent but not very convenient from a practical point of view. Fortunately, the sufficient condition for solvability of the system, with a high practical importance, stated in Theorem 3.3 is valid even for system (28) (see ŠTĚPNIČKA et al. [39], [40]).

*Theorem 3.7:* Let  $A_i$  for  $i = 1, \dots, n$  be normal and fulfill the Ruspini condition (27). Then the system (28) is solvable.

We may easily observe, that each of the important results well known for the interpolativity issue in case of CRI has its analogy even for the BK-Subproduct case under exactly the same conditions and therefore, both inferences are equally appropriate from this point of view.

In fact, the availability of such results for the BK-Subproduct was also one of the motivations to conduct this study of the BK-Subproduct inference *vis-à-vis* the already known advantageous properties of CRI.

### B. Continuity of Fuzzy Inference Systems - Property (ii)

In [28], [30] the authors have dealt with the continuity of a fuzzy function  $f_R^\circ$  adjoint to the CRI mechanism and a fuzzy relation modelling fuzzy rules (3). They have defined

continuity suitably and have shown that it is equivalent to the correctness of the model under consideration.

Although the original definition in [28] of a continuous model was given for the particular inference mechanism CRI, i.e., for  $\circledast \equiv \circ$ , the particular composition plays absolutely no role in the proof of the result (see Theorem 3.11 below) explaining the nature of the definition. Hence can be generalized for an arbitrary fuzzy relational composition.

*Definition 3.8:* A fuzzy relation  $R \in \mathcal{F}(X \times Y)$  is said to be a *continuous* model of fuzzy rules (3) in a structure  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \circledast)$  if for each  $i \in I$  and for each  $A \in \mathcal{F}(X)$  the following inequality holds:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A \circledast R)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \quad (29)$$

*Remark 3.9:* Inequality (29) can be rewritten in terms of the adjoint fuzzy function  $f_R^\circ$  as follows:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (f_R^\circ(A))(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \quad (30)$$

To be precise, the continuity concerns the fuzzy function  $f_R^\circ$ , i.e., the model of fuzzy rules  $R$  as well as the inference mechanism  $\circledast$ . Therefore, we could more adequately talk about continuity of the whole fuzzy inference systems than just about continuity of  $R$ . Nevertheless, since Definition 3.8 fixes a structure in which the continuity is introduced, it is mathematically correct and we adopt the original terminology from [28].

*Remark 3.10:* Let us explain why formula (29) expresses the continuity. The closeness between fuzzy sets is measured by the biresiduation operation  $\leftrightarrow$ , i.e., it is a dual concept to the metric one. Let us consider a continuous Archimedean t-norm  $*$  with an additive generator  $g : [0, 1] \rightarrow [0, +\infty]$ . Then the biresiduum may be written in the form

$$a \leftrightarrow b = g^{-1}(|g(a) - g(b)|), \quad (31)$$

where  $g^{-1} : [0, \infty] \rightarrow [0, 1]$  is the inverse function and where in the case of  $g(0) = \infty$  we define  $g(0) - g(0) = 0$ . Now, for an arbitrary non-empty universe  $X$  it is possible to define a metric  $D_g$  on  $\mathcal{F}(X)$  generated by  $g$  as follows:

$$D_g(A, B) = \bigvee_{x \in X} |g(A(x)) - g(B(x))|. \quad (32)$$

The following theorem justifies the use of the notion of continuity in Definition 3.8.

*Theorem 3.11:* Let  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \circledast)$  be a structure for fuzzy rules (3) such that  $\mathcal{L}$  be a residuated lattice on  $[0, 1]$  with a continuous Archimedean t-norm  $*$  having a continuous additive generator  $g$ . A fuzzy relation  $R \in \mathcal{F}(X \times Y)$  is a continuous model of the fuzzy rules in the given structure  $S$  if and only if

$$D_g(B_i, (A \circledast R)) \leq D_g(A_i, A), \quad i = 1, \dots, n \quad (33)$$

for each fuzzy set  $A \in \mathcal{F}(X)$ .

*Proof:* The proof for  $\circledast \equiv \circ$  may be found in [28]. Its generalization is straightforward.  $\blacksquare$

The main result in PERFILIEVA et al. [28], [30] concerns the relationship of the above mentioned continuity and the interpolativity for the CRI as contained in the following result.

*Theorem 3.12 ([28], Theorem 2):* Let

$S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \circ)$  be a structure for fuzzy rules (3). A fuzzy relation  $R \in \mathcal{F}(X \times Y)$  is a *correct* model of fuzzy rules (3) in the given structure  $S$  if and only if it is a continuous model of these rules in  $S$ .

In the following, we show that an identical result is valid even for the case of the BK-Subproduct. Let us start by proving the following lemma which is crucial for further results.

*Lemma 3.13:* Let  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \triangleleft)$  be a structure for fuzzy rules (3) and let  $R \in \mathcal{F}(X \times Y)$ . Then for any  $A \in \mathcal{F}(X)$  and all  $i = 1, \dots, n$  and  $y \in Y$  it is true that

$$B_i(y) \leftrightarrow (A \triangleleft R)(y) \geq \delta_{R,i}(y) * \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)), \quad (34)$$

where  $\delta_{R,i}(y) = B_i(y) \leftrightarrow (A_i \triangleleft R)(y)$ .

*Proof:* By the transitivity property (21) of  $\leftrightarrow$  with respect to  $*$ , we get

$$\begin{aligned} & B_i(y) \leftrightarrow (A \triangleleft R)(y) \\ & \geq (B_i(y) \leftrightarrow (A_i \triangleleft R)(y)) * ((A_i \triangleleft R)(y) \leftrightarrow (A \triangleleft R)(y)) \end{aligned}$$

where  $i \in \{1, \dots, n\}$ .

The first multiplicand  $B_i(y) \leftrightarrow (A_i \triangleleft R)(y)$  is equal to  $\delta_{R,i}(y)$ , while the second multiplicand, for an arbitrary  $y \in Y$ , has the following lower bound:

$$\begin{aligned} & (A \triangleleft R)(y) \leftrightarrow (A_i \triangleleft R)(y) \\ & = \bigwedge_{x \in X} (A(x) \rightarrow R(x, y)) \leftrightarrow \bigwedge_{x \in X} (A_i(x) \rightarrow R(x, y)) \\ & \geq \bigwedge_{x \in X} ((A(x) \rightarrow R(x, y)) \leftrightarrow (A_i(x) \rightarrow R(x, y))) \quad [\text{By (22)}] \\ & = \bigwedge_{x \in X} \left( ([A(x) \rightarrow R(x, y)] \rightarrow [A_i(x) \rightarrow R(x, y)]) \right. \\ & \quad \left. \wedge ([A_i(x) \rightarrow R(x, y)] \rightarrow [A(x) \rightarrow R(x, y)]) \right) \\ & = \bigwedge_{x \in X} \left( ([A_i(x) * (A(x) \rightarrow R(x, y))] \rightarrow R(x, y)) \right. \\ & \quad \left. \wedge ([A(x) * (A_i(x) \rightarrow R(x, y))] \rightarrow R(x, y)) \right) \quad [\text{By (14)}] \\ & = \bigwedge_{x \in X} \left( (A_i(x) \rightarrow [(A(x) \rightarrow R(x, y)) \rightarrow R(x, y)]) \right. \\ & \quad \left. \wedge (A(x) \rightarrow [(A_i(x) \rightarrow R(x, y)) \rightarrow R(x, y)]) \right) \quad [\text{By (14)}] \\ & \geq \bigwedge_{x \in X} \left( [A_i(x) \rightarrow A(x)] \wedge [A(x) \rightarrow A_i(x)] \right) \quad [\text{By (15)}] \\ & = \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)). \end{aligned}$$

Now (34) follows immediately from the monotonicity of  $*$ . ■

Due to Lemma 3.13, we may prove the following theorem to which is analogous to Theorem 3.12. It again shows that the

BK subproduct as an inference mechanism carries the same property as the CRI.

*Theorem 3.14:* Let  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \triangleleft)$  be a structure for fuzzy rules (3). A fuzzy relation  $R \in \mathcal{F}(X \times Y)$  is a correct model of fuzzy rules (3) in the given structure  $S$  if and only if it is a continuous model of these rules in  $S$ .

*Proof:* Suppose that  $R$  is a correct model of the fuzzy rules (3) in the given structure. Then  $R$  solves the given system of fuzzy relation equations  $A_i \triangleleft R = B_i$  for all  $i = 1, \dots, n$  and therefore  $\delta_{R,i}(y) = 1$  for all  $i = 1, \dots, n$  and for all  $y \in Y$ . By (34),  $R$  is a continuous model.

Conversely, let  $R$  be a continuous model of (3) in the given structure. Then

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A \triangleleft R)(y)) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x)) \quad (35)$$

holds for each  $i = 1, \dots, n$  and for arbitrary  $A \in \mathcal{F}(X)$ . Substituting  $A \equiv A_i$  in (35), we obtain

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (A_i \triangleleft R)(y)) \geq 1,$$

which implies that  $A_i \triangleleft R \equiv B_i$ . ■

As in the case of interpolativity, the continuity property is present in both the types of inferences under exactly the same conditions.

### C. Robustness of Fuzzy Inference Systems - Property (iii)

Let  $X$  be a classical set and let  $\sim$  be an equivalence relation defined on  $X$ , i.e.,  $\sim$  is reflexive, symmetric and transitive. Immediately,  $\sim$  partitions  $X$  into equivalence classes. It is well-known then that an  $M \subseteq X$  belongs to this partition if, and only if, whenever  $x \in M$  and  $x \sim y$  for some  $y \in X$  then  $y \in M$ . In a sense, the elements of  $M$  are indistinguishable and can be represented mathematically as follows:

$$x \in M \text{ and } x \sim y \text{ implies } y \in M.$$

A similar relation between fuzzy equivalence relations and fuzzy sets on  $X$  was introduced by KŁAWONN and CASTRO [31]. The operation  $*$  comes from the residuated lattice  $\mathcal{L}$ .

*Definition 3.15:* A fuzzy subset  $E$  of the Cartesian product  $X^2$  is called a *fuzzy equivalence relation* on  $X$  if the following properties are satisfied for all  $x, y, z \in X$ :

$$\begin{aligned} (\text{Reflexivity}) \quad & E(x, x) = 1, & (\text{ER}) \\ (\text{Symmetry}) \quad & E(x, y) = E(y, x), & (\text{ES}) \\ (\text{Transitivity}) \quad & E(x, z) \geq E(x, y) * E(y, z). & (\text{ET}) \end{aligned}$$

*Definition 3.16:* A fuzzy set  $\mu \in \mathcal{F}(X)$  is called *extensional* with respect to a fuzzy equivalence relation  $E$  on  $X$  if

$$\mu(x) * E(x, y) \leq \mu(y), \quad x, y \in X. \quad (36)$$

If a fuzzy set  $\mu$  is not extensional with respect to the considered fuzzy equivalence relation  $E$ , instead one considers the smallest fuzzy set that is extensional with respect to  $E$  and contains  $\mu$ .

*Definition 3.17:* Let  $\mu \in \mathcal{F}(X)$  and let  $E$  be a fuzzy equivalence relation on  $X$ . The fuzzy set

$$\widehat{\mu}(x) = \bigwedge \{ \nu \mid \mu \leq \nu \text{ and } \nu \text{ is extensional with respect to } E \}$$

is called the *extensional hull* of  $\mu$ . Note that by  $\mu \leq \nu$  we mean that for all  $x \in X$ ,  $\mu(x) \leq \nu(x)$ , i.e., we mean ordering in the sense of inclusion, not in the sense of ordering fuzzy quantities.

*Proposition 3.18 ([31], Proposition 2.9):* Let  $\mu \in \mathcal{F}(X)$  and let  $E$  be a fuzzy equivalence relation on  $X$ . Then

- (i)  $\widehat{\mu}(x) = \bigvee \{ \mu(y) * E(x, y) \mid y \in X \}$ ,
- (ii)  $\widehat{\mu}$  is extensional with respect to  $E$ ,
- (iii)  $\widehat{\widehat{\mu}} = \widehat{\mu}$ .

The following two important and interesting results about the CRI scheme and the indistinguishability inherent to the fuzzy sets considered are proved in [31].

*Theorem 3.19 ([31], Theorem 4.4):* Let

$S = (X, Y, \{A, B\}, \mathcal{L}, \circ)$  be a structure for a single fuzzy IF-THEN rule as given in (1). Let  $E$  be a fuzzy equivalence relation on  $X$  with respect to which  $A$  is extensional. Let  $A' \in \mathcal{F}(X)$  be any fuzzy set, then

$$\begin{aligned} A' \circ \widehat{R} &= \widehat{A'} \circ \widehat{R}, \\ A' \circ \check{R} &= \widehat{A'} \circ \check{R}. \end{aligned}$$

The following interpretation of the above result is given in [31]: *The output obtained from CRI for a given fuzzy rule and an input fuzzy set  $A'$  does not change if we substitute  $A'$  by its extensional hull  $\widehat{A'}$ . The indistinguishability inherent in the fuzzy set  $A$  cannot be avoided even if the input fuzzy set  $A'$  stands for a crisp value. Further, a fuzzified input does not change the outcome of a rule as long as the fuzzy set obtained by the fuzzification is contained in the extensional hull of the original crisp input value. They finally conclude that it does not make sense to measure more exactly than the indistinguishability admits.*

In other words, this shows the robustness of the inference in scenarios where there can be slight discrepancies between the intended input and the actual input.

It is immediate now, as already observed in [31], that *the indistinguishability induced by the fuzzy set representing the linguistic expression in the premise of the rule cannot be overcome.*

Even though Theorem 3.19 is proven in [31] for a single fuzzy rule, it can be shown that the result is valid even with  $n$  fuzzy rules.

*Theorem 3.20:* Let  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \circ)$  be a structure for fuzzy rules (3). Let  $E$  be a fuzzy equivalence relation on  $X$  with respect to which  $A_i$  is extensional for arbitrary  $i = 1, \dots, n$ . Let  $A' \in \mathcal{F}(X)$  be any fuzzy set, then  $A' \circ \widehat{R} = \widehat{A'} \circ \widehat{R}$ .

*Proof:* The inequality  $\widehat{A'} \circ \widehat{R} \geq A' \circ \widehat{R}$  holds for arbitrary fuzzy relation  $R \in \mathcal{F}(X \times Y)$ .

Thus it suffices to prove the other inequality for  $\check{R}$ , i.e.,  $\widehat{A'} \circ \check{R} \leq A' \circ \check{R}$ . Since  $A_i$  is extensional with respect to  $E$  for arbitrary  $i$ ,  $A_i(x') \geq A_i(x) * E(x, x')$  for any  $x, x' \in X$ .

For any  $x \in X$ , we have

$$\begin{aligned} \widehat{A'}(x) * \bigvee_{i=1}^n (A_i(x) * B_i(y)) &= \bigvee_{x' \in X} \left( A'(x') * E(x, x') \right) * \bigvee_{i=1}^n \left( A_i(x) * B_i(y) \right) \\ &= \bigvee_{x' \in X} \left( [A'(x') * E(x, x')] * \bigvee_{i=1}^n (A_i(x) * B_i(y)) \right) \\ &= \bigvee_{i=1}^n \bigvee_{x' \in X} \left( [A'(x') * E(x, x')] * (A_i(x) * B_i(y)) \right) \\ &= \bigvee_{i=1}^n \bigvee_{x' \in X} \left( A'(x') * [E(x, x') * A_i(x)] * B_i(y) \right) \\ &\leq \bigvee_{i=1}^n \bigvee_{x' \in X} \left( A'(x') * (A_i(x') * B_i(y)) \right), \end{aligned}$$

which implies

$$(\widehat{A'} \circ \check{R})(y) \leq (A' \circ \check{R})(y), \quad y \in Y.$$

■

It should be emphasized that only the Mamdani-Assilian (Cartesian product) approach  $\check{R}$  generally works in the combination with the CRI. We now show the robustness of the BK-Subproduct inference mechanism along similar lines as KLAWONN and CASTRO [31]. Once again the employed operations come from the residuated lattice  $\mathcal{L}$ . Firstly note that if a fuzzy set  $\mu \in \mathcal{F}(X)$  is extensional with respect to a fuzzy equivalence relation  $E$  on  $X$  then

$$E(x, y) \rightarrow \mu(y) \geq \mu(x), \quad x, y \in X. \quad (37)$$

*Proposition 3.21:* Let  $\mu \in \mathcal{F}(X)$  and  $E$  a fuzzy equivalence relation on  $X$ . Then

$$\widehat{\mu}(x) = \bigwedge \{ E(x, y) \rightarrow \mu(y) \mid y \in X \}. \quad (38)$$

*Proof:* Let  $\tilde{\mu}(y) = \bigwedge \{ E(z, y) \rightarrow \mu(z) \mid z \in X \}$ . We only need to show that  $\tilde{\mu} = \widehat{\mu}$ . Note, firstly, that for any  $x \in X$  we have

$$\begin{aligned} \tilde{\mu}(x) &= \bigwedge \{ E(z, x) \rightarrow \mu(z) \mid z \in X \} \leq E(x, x) \rightarrow \mu(x) \\ &= 1 \rightarrow \mu(x) = \mu(x) \leq \widehat{\mu}(x). \end{aligned}$$

Let  $\nu \in \mathcal{F}(X)$  be extensional with respect to  $E$  such that  $\nu \geq \mu$ , which implies, by definition, that  $\widehat{\nu} \leq \nu$ . Then for any  $x \in X$  we have

$$\nu(x) \leq E(z, x) \rightarrow \nu(z) \quad \text{and} \quad \nu(x) \leq E(z, x) \rightarrow \mu(z),$$

for every  $z \in X$  and therefore

$$\nu(x) \leq \bigwedge \{ E(z, x) \rightarrow \mu(z) \mid z \in X \} = \tilde{\mu}(x),$$

i.e.,  $\tilde{\mu}(x) \geq \widehat{\mu}(x)$  and hence  $\tilde{\mu}(x) = \widehat{\mu}(x)$ . ■

Now, we present a result analogous to the one given in Theorem 3.19.

*Theorem 3.22:* Let  $S = (X, Y, \{A, B\}, \mathcal{L}, \triangleleft)$  be a structure for fuzzy rule (1). Let  $E$  be a fuzzy equivalence relation on  $X$  with respect to which  $A$  is extensional. Let  $A' \in \mathcal{F}(X)$  be any fuzzy set, then

$$A' \triangleleft \hat{R} = \widehat{A'} \triangleleft \hat{R}, \quad A' \triangleleft \check{R} = \widehat{A'} \triangleleft \check{R}.$$

*Proof:* Let  $R \in \mathcal{F}(X \times Y)$  be any fuzzy relation. By the definition of  $\widehat{A'}$  we have the following inequalities:

$$\begin{aligned} \widehat{A'} \geq A' &\implies \widehat{A'} \rightarrow R \leq A' \rightarrow R \quad \text{[By (12)]} \\ &\implies \widehat{A'} \triangleleft R \leq A' \triangleleft R. \end{aligned}$$

Thus it suffices to prove the other inequality, i.e.,  $\widehat{A'} \triangleleft R \geq A' \triangleleft R$ .

Let  $R = \hat{R}$ , i.e.,  $R(x, y) = A(x) \rightarrow B(y)$  for any  $x \in X, y \in Y$ . Then, by definition, we have

$$(\widehat{A'} \triangleleft \hat{R})(y) = \bigwedge_{x \in X} (\widehat{A'}(x) \rightarrow (A(x) \rightarrow B(y))), \quad y \in Y.$$

Since  $A$  is extensional with respect to  $E$ ,  $A(x') \geq A(x) * E(x, x')$  for any  $x, x' \in X$  and by (12)

$$A(x') \rightarrow B(y) \leq (A(x) * E(x, x')) \rightarrow B(y), \quad y \in Y. \quad (39)$$

For any  $x \in X$ , we have

$$\begin{aligned} \widehat{A'}(x) \rightarrow (A(x) \rightarrow B(y)) &= \bigvee_{x' \in X} \left( A'(x') * E(x, x') \right) \rightarrow (A(x) \rightarrow B(y)) \\ &= \bigwedge_{x' \in X} \left( [A'(x') * E(x, x')] \rightarrow (A(x) \rightarrow B(y)) \right) \quad \text{[By (18)]} \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow [E(x, x') \rightarrow (A(x) \rightarrow B(y))] \right) \quad \text{[By (14)]} \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow ((E(x, x') * A(x)) \rightarrow B(y)) \right) \quad \text{[By (14)]} \\ &\geq \bigwedge_{x' \in X} \left( A'(x') \rightarrow (A(x') \rightarrow B(y)) \right) \quad \text{[By (39)]} \end{aligned}$$

which implies

$$\begin{aligned} (\widehat{A'} \triangleleft \hat{R})(y) &= \bigwedge_{x \in X} (\widehat{A'}(x) \rightarrow (A(x) \rightarrow B(y))) \\ &\geq \bigwedge_{x' \in X} \left( A'(x') \rightarrow (A(x') \rightarrow B(y)) \right) = (A' \triangleleft \hat{R})(y), \end{aligned}$$

for any  $y \in Y$ .

Let  $R = \check{R}$ , i.e.,  $R(x, y) = A(x) * B(y)$  for any  $x \in X, y \in Y$ . Then, by definition, we have

$$(\widehat{A'} \triangleleft \check{R})(y) = \bigwedge_{x \in X} (\widehat{A'}(x) \rightarrow (A(x) * B(y))), \quad y \in Y.$$

For any  $x \in X$ , we have

$$\begin{aligned} \widehat{A'}(x) \rightarrow (A(x) * B(y)) &= \bigvee_{x' \in X} \left( A'(x') * E(x, x') \right) \rightarrow (A(x) * B(y)) \\ &= \bigwedge_{x' \in X} \left( (A'(x') * E(x, x')) \rightarrow (A(x) * B(y)) \right) \quad \text{[By (18)]} \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow [E(x, x') \rightarrow (A(x) * B(y))] \right) \quad \text{[By (14)]} \end{aligned}$$

which is by (19,13)

$$\begin{aligned} &\geq \bigwedge_{x' \in X} \left( A'(x') \rightarrow ((E(x, x') \rightarrow A(x)) * B(y)) \right) \\ &\text{and by (16)} \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow \bigwedge_{x \in X} \left( (E(x, x') \rightarrow A(x)) * B(y) \right) \right) \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow \left( \bigwedge_{x \in X} (E(x, x') \rightarrow A(x)) * B(y) \right) \right) \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow (\widehat{A'}(x') * B(y)) \right) \quad \text{[By (38)]} \\ &= \bigwedge_{x' \in X} \left( A'(x') \rightarrow (A(x') * B(y)) \right) \quad \text{[}\because \widehat{A} = A, \end{aligned}$$

which implies that  $(\widehat{A'} \triangleleft \check{R})(y) = (A' \triangleleft \check{R})(y)$ , for any  $y \in Y$ .  $\blacksquare$

The above result, as already noted in the case of CRI, shows the robustness of the BK-Subproduct inference in scenarios where there can be slight discrepancies between the intended input and the actual input and reinforces the fact that even in the case of BK-Subproduct the indistinguishability induced by the fuzzy set representing the linguistic expression in the premise of the rule cannot be overcome.

Once again, as in the case of CRI, we may generalize the result concerning the indistinguishability of the premises for an arbitrary finite number of rules. Note that in the case of the BK-Subproduct the fuzzy relation  $\hat{R}$  is plays the main role.

*Theorem 3.23:* Let  $S = (X, Y, \{A_i, B_i\}_{i=1, \dots, n}, \mathcal{L}, \triangleleft)$  be a structure for fuzzy rules (3). Let  $E$  be a fuzzy equivalence relation on  $X$  with respect to which each  $A_i$  is extensional, for arbitrary  $i = 1, \dots, n$ . Let  $A' \in \mathcal{F}(X)$  be any fuzzy set, then

$$A' \triangleleft \hat{R} = \widehat{A'} \triangleleft \hat{R}.$$

*Proof:* The inequality  $\widehat{A'} \triangleleft R \leq A' \triangleleft R$  holds for arbitrary fuzzy relation  $R \in \mathcal{F}(X \times Y)$ , see the proof of Theorem 3.22.

Thus it suffices to prove the other inequality for  $\hat{R}$ , i.e.,  $\widehat{A'} \triangleleft \hat{R} \geq A' \triangleleft \hat{R}$ . Since each  $A_i$  is extensional with respect to  $E$  for arbitrary  $i$ ,  $A_i(x') \geq A_i(x) * E(x, x')$  for any  $x, x' \in X$  and we have by (12)

$$\begin{aligned} A_i(x') \rightarrow B_i(y) &\leq (A_i(x) * E(x, x')) \rightarrow B_i(y), \quad y \in Y. \quad (40) \end{aligned}$$

For any  $x \in X$ , we have

$$\begin{aligned}
\widehat{A'}(x) &\rightarrow \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)) \\
&= \bigvee_{x' \in X} \left( A'(x') * E(x, x') \right) \rightarrow \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)) \\
&\text{which is by (18,16,14,40)} \\
&= \bigwedge_{x' \in X} \left( [A'(x') * E(x, x')] \rightarrow \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)) \right) \\
&= \bigwedge_{i=1}^n \bigwedge_{x' \in X} \left( [A'(x') * E(x, x')] \rightarrow (A_i(x) \rightarrow B_i(y)) \right) \\
&= \bigwedge_{i=1}^n \bigwedge_{x' \in X} \left( A'(x') \rightarrow [E(x, x') \rightarrow (A_i(x) \rightarrow B_i(y))] \right) \\
&= \bigwedge_{i=1}^n \bigwedge_{x' \in X} \left( A'(x') \rightarrow ((E(x, x') * A_i(x)) \rightarrow B_i(y)) \right) \\
&\geq \bigwedge_{i=1}^n \bigwedge_{x' \in X} \left( A'(x') \rightarrow (A_i(x') \rightarrow B_i(y)) \right)
\end{aligned}$$

which implies  $(\widehat{A'} \triangleleft \widehat{R})(y) \geq (A' \triangleleft \widehat{R})(y)$ ,  $y \in Y$ .  $\blacksquare$

#### IV. COMPUTATIONAL ASPECTS OF FUZZY RELATIONAL INFERENCE

In this section, we deal with the computational aspects of CRI and BK-Subproduct inferences. Firstly, we show that all the advantages enjoyed by CRI are also available with the BK-Subproduct too. However, both CRI and BK-Subproduct - being fuzzy relational inferences - possess some drawbacks. Recently JAYARAM [41] had proposed a modified form of CRI, viz., Hierarchical CRI scheme, to overcome some of these drawbacks. We show that a similar hierarchical inferencing is possible even in the case of BK-Subproduct and hence is a computationally viable alternative for the CRI.

##### A. Inferencing in the Case of Singleton Fuzzy Inputs - Property (iv)

The following definition will be useful in this subsection.

*Definition 4.1:* A fuzzy set on a non-empty set  $X$ ,  $A : X \rightarrow [0, 1]$ , is said to be a "fuzzy singleton" if there exists an  $x_0 \in X$  such that  $A$  has the following representation:

$$A(x) = \begin{cases} 1, & \text{if } x = x_0, \\ 0, & \text{if } x \neq x_0. \end{cases} \quad (41)$$

We say  $A$  attains normality at  $x_0 \in X$ .

It is not uncommon in some contexts to deal with fuzzy singleton inputs. For instance, in typical control situations the input is usually a crisp value which is fuzzified before it is presented to a fuzzy system to obtain the output. There are many fuzzification methods, i.e., procedures to convert a crisp value into a fuzzy set with different shapes and spread based on this value. Most often the *singleton fuzzifier* which converts

a crisp input  $x' \in X$  into a singleton  $A' \in \mathcal{F}(X)$  which attains normality at  $x'$  is used. Note, that this seeming formal conversion is crucial since it allows us to apply any fuzzy relational inference which, in principle, deals only with fuzzy inputs.

Given a crisp input  $x' \in X$  and which is fuzzified using a singleton fuzzifier, from a computational point of view, it is highly desirable to deal with such fuzzy relational inference mechanisms whose inferred output is dependent only on the chosen fuzzy relation  $R$  modeling a given fuzzy rule base and the inference plays a role only in case of a fuzzy input, i.e., the inferred output  $f_R^{\circledast}(A') = B' \in \mathcal{F}(Y)$  is given by  $B'(y) = R(x', y)$ , for arbitrary  $y \in Y$ .

This property, which saves computational costs, holds for CRI. From the following equalities, we see that the discussed property is valid even for the BK-Subproduct. Let the given singleton fuzzy input  $A'$  attains normality at some  $x' \in X$ . Then the inferred output using the BK-Subproduct is given by

$$\begin{aligned}
B'(y) &= \bigwedge_{x \in X} (A'(x) \rightarrow R(x, y)) \\
&= (A'(x') \rightarrow R(x', y)) \wedge \bigwedge_{\substack{x \in X \\ x \neq x'}} (A'(x) \rightarrow R(x, y)) \\
&= (1 \rightarrow R(x', y)) \wedge \bigwedge_{\substack{x \in X \\ x \neq x'}} (0 \rightarrow R(x, y)) \\
&= R(x', y) \wedge 1 = R(x', y), \quad y \in Y.
\end{aligned}$$

It should be emphasized that the above property is not generally valid for any fuzzy relational composition. Neither the Bandler-Kohout superproduct nor the Bandler-Kohout square product [24], [42] nor any of the  $\inf - S$  fuzzy relational compositions keep this essential property. This is one of the reasons why the BK-Subproduct is, other than the CRI, a privileged composition and gives a clear motivation to study all the properties investigated in Section III above.

##### B. Equivalence between FITA and FATI - Property (v)

It is a well known fact that one of the reasons to use the Cartesian product approach to model a fuzzy rule base is the possible saving of computational efforts. In other words, combination of  $\circ$  with  $\tilde{R}$  requires fewer computations than the combination with the fuzzy relation  $\hat{R}$ . This is due to the following sequence of equalities:

$$\begin{aligned}
B(y) &= \bigvee_{x \in X} \left( A'(x) * \bigvee_{i=1}^n (A_i(x) * B_i(y)) \right) \\
&= \bigvee_{x \in X} \bigvee_{i=1}^n (A'(x) * (A_i(x) * B_i(y))) \\
&= \bigvee_{i=1}^n \bigvee_{x \in X} ((A'(x) * A_i(x)) * B_i(y)) \\
&= \bigvee_{i=1}^n \left( \bigvee_{x \in X} (A'(x) * A_i(x)) * B_i(y) \right), \quad y \in Y.
\end{aligned}$$

It means that we do not have to compose all rules to a fuzzy relation, we just find the highest degree of intersection of a given input  $A'$  and a particular rule antecedent and multiply it by the corresponding consequent. This approach is then applied rule per rule and the results are composed together by the maximum operation. This reduction in the computational costs is nothing but the effect of the equivalence of FITA and FATI inference strategies [2].

So, it may be again generally considered even for other inference mechanisms. Due to the following sequence of equalities:

$$\begin{aligned}
B(y) &= \bigwedge_{x \in X} \left( A'(x) \rightarrow \bigwedge_{i=1}^n (A_i(x) \rightarrow B_i(y)) \right) \\
&= \bigwedge_{x \in X} \bigwedge_{i=1}^n (A'(x) \rightarrow (A_i(x) \rightarrow B_i(y))) \\
&= \bigwedge_{i=1}^n \bigwedge_{x \in X} ((A'(x) * A_i(x)) \rightarrow B_i(y)) \\
&= \bigwedge_{i=1}^n \left( \bigvee_{x \in X} (A'(x) * A_i(x)) \rightarrow B_i(y) \right), \quad y \in Y,
\end{aligned}$$

we may state that even in the case of the BK-Subproduct  $\triangleleft$  and an appropriate model of fuzzy rules (3) given by  $\tilde{R}$ , the FATI inference strategy is equivalent to the FITA inference strategy.

The difference lies in the propriety of the chosen fuzzy relation modelling fuzzy rules (3) with respect to a chosen inference mechanism. In the case of CRI, there is no other choice but  $\tilde{R}$  if we want to reduce the computational efforts by an equivalent FITA strategy. While in case of the BK subproduct, the same holds for  $\tilde{R}$ . In other words, there should be other reasons leading to the use of  $\tilde{R}$  than the computational costs because if this is the only reason, we still may reduce the computational efforts by the use of the BK subproduct as an inference mechanism while still keeping the conditional nature of rules (3) by the choice of  $\tilde{R}$ . Then the correctness of the model (fundamental interpolation condition) should be ensured by the conditions given in Theorem 3.6. On the other hand, this is also the case when  $\tilde{R}$  is employed where the same conditions have to be imposed, see Theorem 3.2.

*Remark 4.2:* Note that the expression  $\bigvee_{x \in X} (A(x) * A'(x))$ , with  $*$  a t-norm, is in fact, one of the earliest measures proposed by ZADEH [43] to determine the similarity between two fuzzy sets  $A, A' \in \mathcal{F}(X)$ . Moreover, typically, antecedent fuzzy sets form some partition (e.g. the Ruspini partition [44]) and the input fuzzy set is of a limited support so that there are such  $i$  for which the similarity between the input  $A'$  and the antecedent  $A_i$ , given by the above measure, is zero and the computation gets further simplified.

### C. Drawbacks of a Fuzzy Relational Inference

So far, we have considered only Single-Input-Single-Output (SISO) fuzzy rules. However, in practice, one often encounters situations which demand that inputs from multiple sources /

dimensions be considered and hence the need arises for dealing with Multiple-Input-Single-Output (MISO) fuzzy rules of the following type - for the sake of notational simplicity and ease of understanding we only deal with a 2-input-1-output MISO fuzzy rules, which can be extended in an obvious way to more than 2 input dimensions:

$$\mathbf{IF} \ x \text{ is } A_i \ \mathbf{AND} \ y \text{ is } B_i \ \mathbf{THEN} \ z \text{ is } C_i \quad (42)$$

where  $A_i \in \mathcal{F}(X)$ ,  $B_i \in \mathcal{F}(Y)$  and  $C_i \in \mathcal{F}(Z)$ , respectively.

Note that both in the case of MISO and SISO rules the input fuzzy set(s) can be seen to be a fuzzy set on either a single domain or a Cartesian product of the domains, hence all the results presented so far, although discussed in the framework of SISO rules, are valid even when dealing with MISO fuzzy rules.

Fuzzy relational inference schemes are not without their drawbacks because of the computational and space complexities involved (see, e.g., CORNELIS et al. [45], MARTIN-CLOUAIRE [46], DEMIRLI and TURKSEN [47]). These are compounded greatly, especially, while dealing with MISO fuzzy rules. Since CRI and the BK-Subproduct both belong to the class of fuzzy relational inferences they are not immune to these drawbacks.

The complexity of an inference algorithm stems mainly from two factors:

- (i) *The process of inference itself.* The fuzzy inferencing schemes are generally resource consuming (both memory and time). Many of the inference schemes discretize the underlying domains and hence the process becomes computationally intensive.
- (ii) *The structure, complexity and the number of rules.* Depending on the shape of the underlying fuzzy sets the number of parameters stored and processed varies. Similarly, the manner in which multiple antecedents are combined affects the processing complexity. Also an increase in the number of rules only exacerbates the problem. As the number of input variables and/or input fuzzy sets increases, there is a combinatorial explosion of rules in multiple fuzzy rule based systems.

We illustrate the above through the following example.

*Example 4.3:* Let  $A = [0.9 \ 0.8 \ 0.7 \ 0.7]$ ,  $B = [1 \ 0.6 \ 0.8]$ ,  $C = [0.1 \ 0.1 \ 0.2]$ , denote fuzzy sets defined on, respectively, the following classical sets

$$X = \{x_1, x_2, x_3, x_4\}, \quad Y = \{y_1, y_2, y_3\}, \quad Z = \{z_1, z_2, z_3\}.$$

Let  $S = (X, Y, Z, \{A, B, C\}, \mathcal{L}, \triangleleft)$  be the structure considered for the single fuzzy rule:

$$\mathbf{IF} \ x \text{ is } A \ \mathbf{AND} \ y \text{ is } B \ \mathbf{THEN} \ z \text{ is } C .$$

Let  $\mathcal{L}$  be the Łukasiewicz complete residuated lattice, i.e.,  $\mathcal{L} = ([0, 1], \wedge, \vee, \otimes, \rightarrow_{\otimes}, 0, 1)$  where  $\otimes$  stands for the Łukasiewicz t-norm  $x \otimes y = \max(0, x + y - 1)$  and  $\rightarrow_{\otimes}$  stands for the Łukasiewicz implication  $x \rightarrow_{\otimes} y = \min(1, 1 - x + y)$ .

Now, taking the Cartesian product of  $A$  and  $B$  with respect to  $\otimes$ , we have

$$A \otimes B = \begin{pmatrix} 0.9 & 0.5 & 0.7 \\ 0.8 & 0.4 & 0.6 \\ 0.7 & 0.3 & 0.5 \\ 0.7 & 0.3 & 0.5 \end{pmatrix}.$$

Now, we have  $\hat{R}(A, B; C) = [\hat{R}(z_1) \hat{R}(z_2) \hat{R}(z_3)]$ , where  $(A \otimes B) \rightarrow_{\otimes} C = (A \otimes B) \rightarrow_{\otimes} [0.1 \ 0.1 \ 0.2]$  and  $\hat{R}(z_i) = (A \otimes B) \rightarrow_{\otimes} z_i$ . Thus

$$\hat{R}(z_1) = \hat{R}(z_2) = \begin{pmatrix} 0.2 & 0.6 & 0.4 \\ 0.3 & 0.7 & 0.5 \\ 0.4 & 0.8 & 0.6 \\ 0.4 & 0.8 & 0.6 \end{pmatrix};$$

$$\hat{R}(z_3) = \begin{pmatrix} 0.3 & 0.7 & 0.5 \\ 0.4 & 0.8 & 0.6 \\ 0.5 & 0.9 & 0.7 \\ 0.5 & 0.9 & 0.7 \end{pmatrix}.$$

Let  $A' = [0.7 \ 0.6 \ 0.5 \ 0.5]$ ,  $B' = [0.8 \ 0.5 \ 0.7]$  be the given fuzzy (non-singleton) inputs. Then

$$A' \otimes B' = \begin{pmatrix} 0.5 & 0.1 & 0.4 \\ 0.4 & 0.1 & 0.3 \\ 0.3 & 0 & 0.2 \\ 0.3 & 0 & 0.2 \end{pmatrix}.$$

The output obtained from the BK-Subproduct is

$$C' = (A' \otimes B') \triangleleft ((A \otimes B) \rightarrow_{\otimes} C) = [0.7 \ 0.7 \ 0.8]. \quad (43)$$

*Remark 4.4:* With the help of Example 4.3 above the following observations can be made:

- (i) *Computational complexity:* Though the computational complexity largely depends on the choice of operators employed, let us consider the following general case of a  $p$ -input 1-output system where  $i$ -th fuzzy rule is modelled by the fuzzy relation  $R_i \in \mathcal{F}(X_1, \dots, X_p, Y)$  where  $R_i = ((A_i^1 * \dots * A_i^p) \rightarrow B_i)$ . Let the universe of discourse  $X_j$  be discretized into  $p_j$  points for each  $j = 1, \dots, p$ . Then the complexity of a single inference is proportional to  $\mathcal{O}(\prod_{j=1}^p p_j)$ . If  $p_j = m$ , then it is  $\mathcal{O}(m^p)$ .
- (ii) *Space complexity:* Again, for a  $p$ -input 1-output system we have a  $p$ -dimensional matrix having  $\prod_{j=1}^p p_j$  entries. Hence we need to store  $p$ -dimensional matrices for every fuzzy if-then rule.
- (iii) *Run-time Space Requirements:* For example, consider inferencing with the BK-Subproduct inference scheme (8) in the case of a 2-input fuzzy if-then rule. Let the universes of discourse  $X_1, X_2$  and  $Y$  be discretized into  $m, k, l$  points, respectively. Then the memory requirements of the algorithm are as follows (see also DEMIRLI and TÜRKSEN [47]):  $m \cdot k \cdot l$  for  $(A * B) \rightarrow C$ ,  $m \cdot k$  for combining the given facts  $A' * B'$  and  $l$  for the consequent, where by  $*$  we denote the t-norm used for the Cartesian product. Overall, it is  $m \cdot k \cdot l + m \cdot k + l$ .

In the case  $m = k = l$ , the memory requirements of the algorithm become  $m^3 + m^2 + m$ . Generalizing this, in the case of  $p$ -inputs we have that the memory requirements of the algorithm is  $\mathcal{O}(m^{(p+1)})$ .

There are many works proposing modifications to the classical CRI in an attempt to enhancing the efficiency in its inferencing, see for example, the works by FULLÉR and his group [48], [49], [50] and those of MOSER and NAVARA [51], [52], [53]. In the case when there are more than two antecedents involved in fuzzy inference, RUAN and KERRE [54] have proposed an extension to the classical CRI, wherein starting from a finite number of fuzzy relations of an arbitrary number of variables but having some variables in common, one can infer fuzzy relations among the variables of interest. DEMIRLI and TÜRKSEN [47] proposed a *Rule Break-up* method and showed that rules with two or more independent variables in their premise can be simplified to a number of inferences of rule bases with simple rules (only one variable in their premise). For further modification of this method see [55].

However, no such works exist for the BK-Subproduct to the best of the authors' knowledge. In the following we propose a modified form of BK-Subproduct that alleviates some of the concerns noted above, along the lines of the Hierarchical CRI proposed by JAYARAM [41].

#### D. Hierarchical BK-Subproduct - Property (vi)

In [41], JAYARAM proposed a hierarchical variant of the CRI where observation on particular axes are taken independently and hierarchically and the overall output is deduced after all observations were used in this step-by-step chain procedure. On the contrary to the usual case where a Cartesian product of all observations is computed and such a product serves as the only fuzzy input with a vector variable. We follow this idea with the BK subproduct as well.

##### Procedure for Hierarchical BK-Subproduct

Step 1 FOR  $i = 1$  TO  $n$  DO

- (i) Calculate  $R'_i \in \mathcal{F}(Y \times Z)$ :  $R'_i = B_i \rightarrow C_i$
- (ii) Calculate  $C'_i \in \mathcal{F}(Z)$ :  $C'_i = B \triangleleft R'_i$
- (iii) Calculate  $R''_i \in \mathcal{F}(X \times Z)$ :  $R''_i = A_i \rightarrow C'_i$
- (iv) Calculate  $C''_i \in \mathcal{F}(Z)$ :  $C''_i = A \triangleleft R''_i$

Step 2 AGGREGATE ALL  $C''_i$  BY MINIMUM

We also remark that, although in [41] the author has given the algorithm for the Hierarchical CRI only in the case of inferencing with a single MISO rule, it can be extended in a straight-forward manner to the case of multiple MISO rules, as done here. However, for the sake of simplicity, the following example, which demonstrates the reduction in computational efforts and memory savings is given in the context of a single MISO rule.

*Example 4.5:* Let the fuzzy sets  $A, B, C, A', B'$  be as in Example 4.3 with the same structure  $S$  for the given fuzzy rule. Inferencing with the Hierarchical BK-Subproduct, given the input  $(A', B')$  we have

$$\text{Step 1 (i)} \quad B \rightarrow_{\otimes} C = \begin{pmatrix} 0.1 & 0.1 & 0.2 \\ 0.5 & 0.5 & 0.6 \\ 0.3 & 0.3 & 0.4 \end{pmatrix} \quad (44)$$

$$\begin{aligned} \text{Step 1 (ii)} \quad C' &= B' \triangleleft (B \rightarrow_{\otimes} C) \\ &= [0.8 \ 0.5 \ 0.7] \triangleleft (B \rightarrow_{\otimes} C) \\ &= [0.3 \ 0.3 \ 0.4] \end{aligned}$$

$$\text{Step 1 (iii)} \quad A \rightarrow_{\otimes} C' = \begin{pmatrix} 0.4 & 0.4 & 0.5 \\ 0.5 & 0.5 & 0.6 \\ 0.6 & 0.6 & 0.7 \\ 0.6 & 0.6 & 0.7 \end{pmatrix}$$

$$\begin{aligned} \text{Step 1 (iv)} \quad C'' &= A' \triangleleft (A \rightarrow_{\otimes} C') \\ &= [0.7 \ 0.6 \ 0.5 \ 0.5] \triangleleft (A \rightarrow_{\otimes} C') \\ &= [0.7 \ 0.7 \ 0.8] . \end{aligned} \quad (45)$$

*Remark 4.6:* From the above example it is clear that we can convert a multi-input system to a single-input hierarchical system employing, both employing the BK-Subproduct inference. The effect becomes more pronounced when we have more than two input variables. From the above Example 4.5 it can be noticed that the most memory intensive step in the inference is the calculation of the ‘current’ output fuzzy set (Steps 1 (ii) & (iv)). Once again, considering the case of a  $p$ -input fuzzy rule, if the input universe of discourse  $X_j, j = 1, 2, \dots, p$  are discretized into  $p_j$  points and the output universe of discourse  $Z$  into  $l$  points, respectively, then the memory requirements of this step, and hence of the algorithm itself, can easily be seen to be  $p^* \cdot l + l + p^*$ , where  $p^* = \max_{j=1}^p p_j$ . In the case  $m = p^* = l$  we have the overall memory requirements to be  $2m + m^2$ . It should also be emphasized that the memory requirements are independent of the number of input variables, as can be expected in any hierarchical setting.

Not only does the Example 4.5 illustrate the computational efficiency of Hierarchical BK-Subproduct inference, it also shows that the inference obtained from the original BK-Subproduct is identical to the one obtained from the proposed Hierarchical BK-Subproduct, i.e., (43) =  $C'$  =  $C''$  = (45). The following result shows that this equivalence is always guaranteed under the structure  $S$  considered in this work.

*Theorem 4.7:* Let  $S = (X, Y, Z, \{A_i, B_i, C_i\}_{i=1, \dots, n}, \mathcal{L}, \triangleleft)$  be a structure for fuzzy rules as given in (42) and let  $\hat{R} \in \mathcal{F}(X \times Y \times Z)$  be given by

$$\bigwedge_{i=1}^n ((A_i(x) * B_i(y)) \rightarrow C_i(z)), \quad x \in X, y \in Y, z \in Z .$$

Then for any  $A \in \mathcal{F}(X)$  and  $B \in \mathcal{F}(Y)$  all  $i = 1, \dots, n$  it is true that

$$(A * B) \triangleleft \hat{R} \equiv \bigwedge_{i=1}^n (A \triangleleft (A_i \rightarrow (B \triangleleft (B_i \rightarrow C_i)))) .$$

*Proof:* For an arbitrary  $z \in Z$  we have

$$\begin{aligned} &[(A * B) \triangleleft \hat{R}](z) \\ &= \bigwedge_{x,y} \left( (A(x) * B(y)) \rightarrow \bigwedge_{i=1}^n ((A_i(x) * B_i(y)) \rightarrow C_i(z)) \right) \\ &= \bigwedge_{i=1}^n \bigwedge_{x,y} \left( (A(x) * B(y)) \rightarrow ((A_i(x) * B_i(y)) \rightarrow C_i(z)) \right) \end{aligned}$$

and by triple use of (14) we get

$$\begin{aligned} &= \bigwedge_{i=1}^n \bigwedge_{x,y} \left( (A(x) * A_i(x) * B(y) * B_i(y)) \rightarrow C_i(z) \right) \\ &= \bigwedge_{i=1}^n \bigwedge_x \left( (A(x) * A_i(x)) \rightarrow \bigwedge_y \left( (B(y) * B_i(y)) \rightarrow C_i(z) \right) \right) \end{aligned}$$

which equals to

$$\bigwedge_{i=1}^n \bigwedge_x \left( A(x) \rightarrow (A_i(x) \rightarrow \bigwedge_y (B(y) \rightarrow (B_i(y) \rightarrow C_i(z)))) \right)$$

and therefore

$$[(A * B) \triangleleft \hat{R}](z) = \bigwedge_{i=1}^n (A \triangleleft (A_i \rightarrow (B \triangleleft (B_i \rightarrow C_i)))) .$$

Theorem 4.7 proves the equivalence of outputs obtained from the proposed algorithm of the hierarchical BK inference mechanism and the original BK-Subproduct with two dimensional inputs. Indeed, it may be systematically extended into a case of inputs of an arbitrary finite dimension. ■

Now, we may state the following corollary of Theorem 3.6 and Theorem 4.7, which claims that if condition (26) certifying the solvability of (28) holds than even the proposed Hierarchical BK-Subproduct inference procedure keeps the fundamental interpolation condition fulfilled.

*Corollary 4.8:* Let all the assumptions of Theorem 4.7 be valid. Furthermore, let

$$\bigvee_{i=1}^n (A_i(x) * A_j(x) * B_i(y) * B_j(y)) \leq \bigwedge_{i=1}^n (C_i(z) \leftrightarrow C_j(z))$$

holds for arbitrary  $x \in X, y \in Y, z \in Z$  and for arbitrary  $i, j \in \{1, \dots, n\}$ . Then

$$\bigwedge_{i=1}^n (A_i \triangleleft (A_i \rightarrow (B_i \triangleleft (B_i \rightarrow C_i)))) \equiv C_i .$$

## V. CONCLUSIONS

In this work, after recalling some of the properties that are usually cited in favor of using the Compositional Rule of Inference (CRI) introduced by ZADEH [1], viz., equivalent and reasonable conditions for their solvability, their interpolative properties and the preservation of the indistinguishability that may be inherent in the input fuzzy sets, we have shown that the Bandler-Kohout subproduct introduced in [24] does possess all the above properties and hence is equally suitable for consideration when reasoning with a system of fuzzy

rules. Towards this end some new but equivalent results on indistinguishability operations has also been presented.

Moreover, we show that under certain conditions the equivalence of FITA and FATI can be shown for the Bandler-Kohout subproduct, much like in the case of CRI. After citing some of the main drawbacks of fuzzy relational inferences, we propose a hierarchical inferencing scheme that alleviates many of these in the BK-Subproduct inference. This method is amply illustrated with numerical examples. Finally, we have also shown that if the structure for the considered fuzzy rules is chosen appropriately then the outputs obtained from the Hierarchical BK-Subproduct and the original BK-Subproduct are identical, thus addressing the issues related to computational complexity.

Based on this work, one can see that the BK-Subproduct is as much advantageous as the classical CRI proposed by Zadeh and hence can be employed alternatively in applications. The main difference lies in the fuzzy relation modelling a fuzzy rule base which is combined with a particular inference mechanism. It is shown that some computational advantages of the very popular Mamdani-Assilian approach are valid only in the case of the CRI inference mechanism, while if we use the BK-Subproduct, then many of the advantages of using  $\hat{R}$  are lost and the implicational approach employing the fuzzy relation  $\hat{R}$  assumes this privilege.

Therefore, we conclude that there is another added value to the existence of two inference schemes with the same appropriate properties - the possibility to freely choose between two approaches of modeling a fuzzy rule base. Up to now, the approach denoted by  $\hat{R}$  employing genuine implication was considered a disadvantage because of its computational complexity, although for some problems it is much more suitable [15]. This investigation shows that from the computational point of view there is neither preferable model of a fuzzy rule base nor preferable inference mechanism, there are only preferable combinations of them.

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