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The SIRM Fuzzy Inference System: A Study of its Suitability

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Approval Sheet

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Declaration

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Contents

Outline of The Work

This work is about how to infer from the given knowledge in the form of various rules. This work studies all the fuzzy inference models that are used widely in almost every application. Moreover, the crux of this work is SIRM inference model. The outline of the work is as follows:

- Firstly, it introduces all the terms which are going to be used throughout this work. The second chapter, 'Preliminaries' consists of definitions and basic concepts that are required to get a thorough understanding of the work.
- Secondly, it talks about various fuzzy inference methods like Fuzzy Relational Inference (FRIs) : Compositional Rule Inference (CRI) and Bandler-Kohout Subproduct Inference Method (BKS) , Single Input Connected Type Fuzzy Inference Method (SIC method) and Takagi-Sugeno Inference Method (TSK). In addition to the fuzzy inference models, the third chapter also explains about the five basic properties, that any FIS is expected to have.
- Thirdly, fourth chapter explains SIRM FIS in detail and also gives the available results about the equivalence between SIRM FIS and other FISs.
- Fourthly, as we have mentioned, that in this work we are going to investigate about whether the desirable five basic properties hold true in case of SIRM, in fifth chapter, two properties, interpolativity and monotonicity have been shown holding true in case of SIRM FIS.
- Lastly, chapter 6 concludes the whole text as in this report.

Chapter 1

Introduction

Fuzzy sets were introduced in order to extend the notion of classical set theory. They were introduced to deal with uncertainty and vagueness. Their biggest application was found in fuzzy logic, which is extension of classical logic. Infact, they gained importance because using them human knowledge can be implemented in models and applications by transforming the knowledge into rules or conditional statements consisting of linguistic terms and variables.
Where crisp sets dictate whether an element belongs to the set or not, fuzzy set tells about the extent to which an element belongs to the set, that extent is termed as membership degree and these membership degrees are assigned to the elements of the domain by the associated membership function. Thus, mathematically, while crisp sets can be taken as functions from the domain ,say, X to $\{0, 1\}$, fuzzy sets are functions from to X to [0,1] which gives membership degree to every element in X. Moreover, fuzzy sets are always defined with respect to the context. That is, while defining fuzzy sets, there is always a context that experts keep in mind.
For an example, let us consider a classical set, $X = [4, 7]$. Suppose, we want to get a set of tall people. If we assume that people with their height between $[5.5, 6.2]$ are tall, then its absurd that people with height 5.4 ft. won't be considered as tall, according to the given classical definition of 'tall'. Here is where classical sets fail and we have to define fuzzy set, 'tall' which will give membership degree to every element of the domain. Also, note that, here the fuzzy set 'tall' is defined for the people, but this fuzzy set will completely differ, if we are talking about a tall buiding, or it will also differ, in case we specify the country of which people we are talking about. Because then the context will vary and hence the definition of fuzzy set.
Now consider a general fuzzy rule,

$$\text{If } \tilde{x} \text{ is } A \text{ then } \tilde{y} \text{ is } B.$$

and let $\mathcal{F}(X)$ and $\mathcal{F}(Y)$ be the set of fuzzy sets on X and Y respectively. Given a fuzzy input, A', the corresponding output B' is obtained through the inference mechanism. Thus, an inference mechanism can be seen as a function from $\mathcal{F}(X)$ to $\mathcal{F}(Y)$. The inference method can be chosen to be any of the following methods depending on the case or the nature of the problem: For instance, FRIs like CRI, BKS take and deduce fuzzy sets, whereas, SIRM, TSK and SIC take crisp inputs and give crisp outputs. However, crisp sets can be fuzzified (either by transforming them to singleton sets or defining membership functions on the domain) and correspondingly, fuzzy sets also can be defuzzified.
As we said that that fuzzy inference systems (FISs) are nothing but functions, so there are several properties that one may want this function to satisfy. Hence, out of many properties, there are few desirable properties that an FIS/function should satisfy, and they are,: 1. Interpolativity 2. Continuity, 3.Monotonicity, 4. Robustness, and 5. Universal Approximation.
Among all the FISs mentioned above, SIRM Inference method is the newest. It has many applications because of its several advantages over other FISs (that can be explored in further sections). It will be studied in detail in the coming sections.

1.1 Motivation for Work

Our work is about single Input-Rule Connected FIS (SIRM), which was introduced by Hirota Seki etal. in their work [Yubazaki et al.(1997)Yubazaki, Yi, and Hirota]. It is the newest inference model and is gaining popularity because of its efficient performance in applications (Its merits can be read in further sections). However, the basic properties, mentioned in the previous section, that should hold in case of a FIS, are not studied that much in case of SIRMs. So, because of the uses this FIS has, in our work we are going to study SIRM FIS in detail and will try to investigate whether those hold true in their case as well.

1.2 Our Approach

The motto behind this work is as follows:

- Study SIRM FIS in detail, and
- Probe whether the basic five properties hold true in case of SIRM FIS or not.

This task can be accomplished using either of the following approaches:

- Investigate each of the properties independently for SIRMs, or
- Check under what conditions on the underlying operations, SIRMs become equivalent to some other FIS satisfying those properties so that equivalence between the two will ensure that those properties will hold true in case of SIRM Connected Type FIS as well.

Chapter 2

Preliminaries

Definition 2.0.1. *Given a domain X, a fuzzy set A is characterised by its membership function, $\mu_A : X \to [0,1]$. Then, the fuzzy set will be written as $A = \{(x, \mu_A(x)) \mid x \in X\}$.*

A is said to be normal if there exists an $x \in X$ such that $\mu_A(x) = 1$, else its called as subnormal fuzzy set.

Support, Kernal, Height and ceiling of A are denoted by Supp A, Ker A, Hgt A, Ceil A respectively and are defined as follows :

- *Supp A* $= Cl\{x \in X \ni \mu_A(x) > 0\}$. *(where Cl(*) denotes closure of * and * can be any set.)*
- *Ker A* $= \{x \in X \ni \mu_A(x) = 1\}$.
- *Hgt A* $= \sup\{\mu_A(x) \ni x \in X\}$.
- *Ceil A*$= \{x \in X \ni \mu_A(x) = HgtA\}$.
- *A is said to be bounded if Supp A is a bounded set.*
  Note that for a normal fuzzy set Ker A = Ceil A and Hgt A = 1.

Definition 2.0.2. ***Fuzzifier:*** *A fuzzifier is an important part of the fuzzy logic system. It converts the crisp value to a fuzzy set. Hence, its a function from a crisp set, X to the set of fuzzy sets on X, $\mathcal{F}(X)$. i.e., $\phi : X \to \mathcal{F}(X)$. The most common fuzzifier that is used widely is singleton fuzzy set,*

$$\phi : X \to \mathcal{F}(X)$$

defined as

$$\phi(x) = A_x \in \mathcal{F}(X) \ ,$$

where, $A_x : X \to [0,1]$ is defined as

$$A_x(x') = \begin{cases} 1, & \text{if } x = x' \ , \\ 0, & \text{if } x \neq x' \ . \end{cases}$$

Defuzzifier: *Antonym to fuzzifier, a defuzzifier converts the fuzzy set to a crisp value. Hence, it is a function from the set of fuzzy sets on X, $\mathcal{F}(X)$ to a crisp set, X . For e.g., Mean of maxima method for defuzzification of a fuzzy set A is given as,*

$$y = \frac{\sum_{ceilA} A(x)}{\sum_{ceilA} x}, \qquad \text{if} \sum_{ceilA} x \neq 0$$

Definition 2.0.3. ***Fuzzy Partitions:*** *Let $\mathcal{P}$ be a finite collection of fuzzy sets of X, i.e, $\mathcal{P} = \{A_k\}_{k=1}^{n} \subseteq \mathcal{F}(X)$. P is said to form a fuzzy partition on X if*

$$X \subseteq \bigcup_{i=1}^{n} Supp(A_i)$$

Fuzzy partition is said to be :

- *consistent, if whenever for some* i , $A_i(x) = 1$, *then* $A_j(x) = 0$, *for* $i \neq j$
- *Ruspini if*

$$\sum_{i=1}^{n} A_i(x) = 1 \quad \text{for every } x \in X \ .$$

Definition 2.0.4. ***t-norms and s-norms:*** *A t-norm operator denoted as* $t(x, y)$ *is a function mapping from* $[0, 1] \times [0, 1]$ *to* $[0, 1]$ *that satisfies the following conditions for any* $w, x, y, z \in [0, 1]$ *:*

- $t(0, 0) = 0,\ t(x, 1) = t(1, x) = x$, *(boundary conditions)*
- $t(x, y) \leq t(z, w)$ *if* $x \leq z$ *and* $y \leq w$, *(Monotonic in both the variables)*
- $t(x, y) = t(y, x)$, *(Commutative)*
- $t(x, t(y, z)) = t(t(x, y), z)$. *(Associative)*

An s-norm or t-conorm denoted as $s(x, y)$ *is a function mapping from* $[0, 1] \times [0, 1]$ *to [0,1] that satisfies the following conditions for any* $w, x, y, z \in [0, 1]$ *:*

- $s(1, 1) = 1,\ s(x, 0) = s(0, x) = x$, *(boundary conditions)*
- $s(x, y) \leq s(z, w)$ *if* $x \leq zandy \leq w$, *(Monotonic in both the variables)*
- $s(x, y) = s(y, x)$, *(Commutative)*
- $s(x, s(y, z)) = s(s(x, y), z)$. *(Associative)*

Definition 2.0.5. ***Fuzzy Implication :****A function* $I : [0, 1] \times [0, 1] \to [0, 1]$ *is called a fuzzy implication if it satisfies, for all* $x, x_1, x_2, y, y_1, y_2 \in [0, 1]$, *the following conditions:*

- *if* $x_1 \leq x_2$, *then* $I(x_1, y) \geq I(x_2, y)$, *i.e.,* $I(, y)$ *is decreasing ,*
- *if* $y_1 \leq y_2$, *then* $I(x, y_1) \leq I(x, y_2)$, *i.e.,* $I(x,)$ *is increasing ,*
- $I(0, 0) = 1,\ I(1, 1) = 1,\ I(1, 0) = 0$.

A fuzzy implication $I : [0, 1] \times [0, 1] \to [0, 1]$ *is said to*

- *satisfy the left neutrality property, if*

$$I(1, y) = y,\ y \in [0, 1] \ .$$

- *satisfy the ordering property, if*

$$I(x, y) = 1 \Longleftrightarrow x \leq y \ .$$

- *be a positive fuzzy implication if* $I(x, y) > 0$, *for all* $x, y \in (0, 1]$.

Definition 2.0.6. ***Fuzzy Relation :*** *Let* X *and* Y *be the domain and codomain of the fuzzy inference mechanism, i.e, consider a rule, If* x' *is* A *then* y' *is* B , $A \in \mathcal{F}(X)$ *and* $Y \in \mathcal{F}(Y)$. *Then, a fuzzy relation is the subset of* $X \times Y$. *A fuzzy relation is nothing but a fuzzy set only, i.e., A fuzzy relation* R *is a function*

$$R : X \times Y \to [0, 1]$$

Definition 2.0.7. ***Rulebase*** *: Whole of the fuzzy theory developed as a science to implement human knowledge and somehow work with it. This human knowledge, in any sort of application, is captured in the form of conditional statements/propositions or if-then statements, for example,*

*IF temperature is **HIGH**, THEN, fanspeed is **HIGH**.*

A rulebase is nothing but a collection of rules. There can be two types of rules.

- *Single Input Single Output (SISO) Rules : As the name suggests, such rules takes in single input to process and deduce single output, after inferring. Or, we can say that in a SISO rule, there is only one fuzzy set in its antecedent as well as its consequent. A general SISO rule look like the following, however an example of a SISO rule is given above (example of an if-then statement given above),*

$$IF \widetilde{X} \ is \ A, \ THEN \ \widetilde{Y} \ is \ B.$$

- *Multiple Input Single Output (MISO) Rules : Such rules takes may take more than one inputs to process and deduce single output, after inferring. We can also say that in a MISO rule, there can be more than one fuzzy sets in its antecedent, which may have same or different domains . A general MISO rule look like the following,*

$$IF \ \widetilde{X_1} \ is \ A_1 \ AND \ \widetilde{X_2} \ is \ A_2 \ AND \ ... \ AND \ \widetilde{X_n} \ is \ A_n, \ THEN \ \widetilde{Y} \ is \ B \ .$$

For example,

*IF temperature is **HIGH** AND humidity is **MORE**,*
*THEN fanspeed is **VERY HIGH**.*

A Fuzzy Inference System (FIS), can be seen as a machine, that takes input (crisp/fuzzy), processes it and gives out the desired output. It consists of:

1. A Fuzzifier (in case it requires fuzzy input and we are giving crisp input),
2. A Fuzzy inference Engine (that forms the core intelligence on how to infer),
3. A Fuzzy if-then rule base (that captures the knowledge of the system),
4. A Defuzzifier (in case we want a crisp output).

Definition 2.0.8. *Fuzzy inference system takes care of the processing of the input on the basis of the given rules. It can be taken as a function defined on a crisp set, say X, to another crisp set , say Y or on $\mathcal{F}(X)$, set of all fuzzy sets on X to $\mathcal{F}(Y)$, set of all fuzzy sets on Y, i.e.,*

- $f : X \to Y$ *(crisp output against crisp input, as in case of SIRM, SIC, TSK), or*
- $\tilde{f} : \mathcal{F}(X) \to \mathcal{F}(Y)$ *(fuzzy outputs are generated against fuzzy inputs, as in case of CRI, BKS)*

Chapter 3

Fuzzy Inference Systems

Given the rulebase and the input , the desired output depends completely on the inference mechanism chosen. The Fuzzy Inference Method in turn depends on factors like, partition of the domain and codomain for the inference mapping, aggregating operations, fuzzification and defuzzification techniques, the t-norms and s-norms employed etc. These factors gave rise to a wide variety of fuzzy inference systems which we will study in this section.

3.0.1 Fuzzy Relational Inference

As we mentioned above, that fuzzy inference mechanism can be viewed as a mapping. FRI mechanisms uses a fuzzy relation R to model a given fuzzy rule base. Consider a rule,

$$\text{If } \tilde{x} \text{ is } A, \text{ then } \tilde{y} \text{ is } B. \tag{3.1}$$

The corresponding output B' to the input A' is given by,

$$B' = A'@R,$$

where, $R = \{((x,y),\ R(x,y))\ ,\ |(x,y) \in X \times Y\}$.
On the basis of the composition operator @ and R, FRI's can be classified into two further inference mechanisms:

3.0.2 Compositional Rule of Inference

The compositional rule of inference (CRI) that was provided by Zadeh is one of the earliest FRIs. Here, from the fuzzy IF THEN rule of the form (3.1) , the output B' corresponding to the input A' can be inferred by

$$B'(y) = \bigvee_{x \in X} (A'(x) * R(x,y))$$

Here, $\bigvee$ denotes any s-norm and * denotes any t-norm. We can also infer using a rulebase consisting of several rules. Let us consider the following rule base

$$\text{Rule } i : \text{If } \tilde{x_1} \text{ is } A_i^1,\ \tilde{x_2} \text{ is } A_i^2 \text{ ... } x_n \text{ is } A_i^n \text{ then } \tilde{y_i}\ is\ B_i\ ,$$

where, $\tilde{x_i}$'s are the linguistic variables and A_is are the linguistic values or the fuzzy sets in the input space, $\tilde{y_i}$s are the linguistic variables and B_is are the linguistic values or the fuzzy sets in the output space and $i = 1, 2, \ldots, m$ give m number of rules.
Now, consider the Rule i of the rule base given in (2) and take @ equivalent to product and $\bigvee$ equivalent to sum. The degree of fitness of the input vector $(x'_1, x'_2, ..., x'_n)$ is calculated as:

$$h_i = \prod_{j=1}^{j=n} (A_i^j(x'_j)) \tag{3.2}$$

Then, the corresponding output fuzzy set for i^{th} rule will be given as: $B'_i(y) = h_i.B_i(y)$ and the final output of the rule base will be given by:

$$B'(y) = \sum_{i=1}^{i=m} B'_i(y)\ .$$

3.0.3 Bandler-Kohout Subproduct Inference Method

For a given fuzzy input $A' \in \mathcal{F}(X)$, the fuzzy output $B' \in \mathcal{F}(Y)$ that is obtained by the BK-subproduct inference mechanism is defined as follows:

$$B'(y) = \bigwedge_{x \in X} (A'(x) \to R(x, y))$$

where $\to$ is a residual implication, $\bigwedge$ denotes a conjunction operator and R is the fuzzy relation that models fuzzy rule (3.1).
This is also known as Inf-I composition and the BKS scheme is noted by $B' = A' \triangleleft R$.

3.1 Takagi-Sugeno Inference Model

Rules in the TS inference method are constituted as

$$\text{Rule} - i : \text{If } x_1 \text{ is } A_i^1,\ x_2 \text{ is } A_i^2 \ ...\ x_n \text{ is } A_i^n \text{ then } y = f_i(x_1, x_2, ..., x_n) \qquad (3.3)$$

where, i ranges between 1 and m, $(x_1, x_2, ..., x_n)$ are variables of the antecedent part, $A_i^1, A_i^2 ... A_i^n$ are the antecedent fuzzy sets and $f_i(x_1, x_2, ..., x_n)$ is the function in the consequent. The inference result y' for inputs $x_1', x_2', ..., x_n'$ is given by

$$y' = \frac{\sum_{i=1}^{m} h_i f_i(x_1', x_2', ..., x_n')}{\sum_{i=1}^{m} h_i}$$

where h_i is same as in (3.2) is the degree of the i^{th} rule, and m is the total number of fuzzy rules.

3.2 SIRM Connected-Type Fuzzy Inference Methods

Here, the consequent part of the rule is a constant. The system has n inputs and one output, and each rule module corresponds to one of the n input items and has only the input item in its antecedent.The rules of the functional-type SIRMs method are given as follows:

$$\text{Rules } i : \{\text{If } \tilde{x}_i \text{ is } A_j^i \text{ then } y_i = c_j^i\}_{j=1}^{m_i} \qquad (3.4)$$

where, the Rules-i stands for the i^{th} single input rule module, $\tilde{x}_i$ corresponding to the i^{th}-input item is the sole variable of the antecedent part of the Rules-i, and y_i is the variable of its consequent part. A_j^i and c_j^i are, respectively, the fuzzy set and constant of the j^{th} rule of the Rules-i, where $i = 1, 2, ..., n$, $j = 1, 2, ..., m_i$, and m_i represents the number of rules in the $Rules - i$.
Given an input $\{x_k'\}_{k=1}^{n}$ to the rulebase, the i^{th} component of the input vector, that is x_i' will be the input to Rules-i and the inference result y_i' of the Rules-i is given as follows:

$$y_i' = \frac{\sum_{i=1}^{m_i} h_j^i c_j^i}{\sum_{i=1}^{m_i} h_j^i} \qquad h_j^i = A_j^i(x_i') \qquad (3.5)$$

And the final inference result y' of the functional-type SIRMs method is given by

$$y' = \sum_{i=1}^{n} w_i y_i' \qquad (3.6)$$

where w_i represents the importance degree of each input item x_i , $(i = 1, 2, ..., n)$, that is, the weights given to each module.

3.3 Single Input Connected Fuzzy Inference Method

Given the same rule base as for SIRM's method above, the SIC method also sets up rule modules to each input item. The final inference result of the SIC method is obtained by the weighted average of the degrees of the antecedent part and consequent part of each rule module. Namely, rule modules and degree h_j^i of the antecedent part of the SIC method are given as those of SIRMs method. The final inference result y' is given as follows by using degrees of antecedent part and consequent part from each rule module:

$$y' = \frac{\sum_{i=1}^{n} \sum_{j=1}^{m_i} h_j^i y_j^i}{\sum_{i=1}^{n} \sum_{j=1}^{m_i} h_j^i} \ .$$

3.4 Simplified Fuzzy Inference Method

The simplified fuzzy inference method is a special case of the TS inference method in which the consequent part of the TS inference method is replaced to constant. Following is how rules of the simplified fuzzy inference method are defined:

$$\text{Rule } i : \text{If } \tilde{x_1} \text{ is } A_i^1,\ \tilde{x_2} \text{ is } A_i^2 \ ...\ \tilde{x_n} \text{ is } A_i^m \text{ then } y = y_i.$$

where, y_i is a number and not a fuzzy set. Given the input to the antecedent part, $x_1', x_2', ..., x_n'$, the inference y' will be obtained as follows :

$$y' = \frac{\sum_{i=1}^{m} h_i y_i}{\sum_{i=1}^{m} h_i}\ ,$$

Here also h_i is same as in (3.2).

3.5 Fuzzy Singleton-Type Inference Method

Fuzzy singleton-type inference method (or simplified fuzzy inference method with weight) takes the following fuzzy inference form :

$$\text{Rule } i : \text{If } \tilde{x_1} \text{ is } A_i^1,\ \tilde{x_2} \text{ is } A_i^2 \ ...\ \tilde{x_n} \text{ is } A_i^n \text{ then } y = y_i/w_i$$

where the weight w_i is a real number such that $w_i \geq 0$. The consequence y' to the input vector, $[x_1', x_2', ..., x_n']$, by the fuzzy singleton-type inference method is obtained as follows :

$$y' = \frac{\sum_{i=1}^{m} h_i w_i y_i}{\sum_{i=1}^{m} h_i w_i}$$

where h_i is calculated same as in (3.2) and the product $h_i w_i$ of the fitness h_i and the weight w_i of y_i is regarded as the degree to which y_i is obtained.

Uptill now, we have seen , how the formation of rules, the antecedents and consequents, the composition , the fuzzy relation taken, etc. give rise to so many inference mechanisms. We have read only 7 till now, which are the most popular ones and are used in most of the applications. In the coming sections, we will read about SIRMs Connected Fuzzy Inference Method in detail and will also study about the properties of fuzzy inference systems.

3.6 Properties of FIS

Up till here, we have seen different fuzzy inference mechanisms. It depends on the situation to be modelled, which inference mechanism should be used. Given an input to a fuzzy or crisp rule base, why only these few mechanisms are chosen? How come they became standard? How to judge, whether a mechanism is efficient or not, or useful or not, or is good enough to implement the information that a modeller has, and aquire the desired results? Fuzzy inference is a function that takes in input, infers it and gives the desired output. To answer above raised questions, some properties were defined that are taken as parameter to determine the efficiency or goodness of an inference mechanism. They are, interpolativity, continuity, monotonicity, robustness, equivalence between FITA (First infer then aggregate) and FATI (First aggregate then infer) and universal approximation. We will study each one by one :

3.6.1 Interpolativity

Interpolativity deals with getting the right output corresponding to the given input. To understand it mathematically, let

- $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$ be the n-dimensional input domain,
- $\mathcal{V}$ be the output domain,
- $\mathcal{D} = \{(\bar{\alpha}_k\ ,\ \beta_k) \in \mathcal{U} \times \mathcal{V}\}_{k=1}^{\ell}$ be the set of data points given about the system,
- where $\bar{\alpha}_k = (\alpha_{k1}, \ldots, \alpha_{kn}) \in \mathcal{U}$.

If using the above information, a rule base is constructed, then a fuzzy inference system will be interpolative if it will infer β_k only, corresponding to the input vector, $\bar{\alpha}_k = (\alpha_{k1}, \ldots, \alpha_{kn})$.

3.7 Continuity

Let us consider a fuzzy rule base as follows:

$$\text{IF } \tilde{x} \text{ is } A_i \text{ THEN } \tilde{y} \text{ is } B_i, \text{ for } i = 1, \ldots, n \tag{3.7}$$

A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is said to be a continuous model of fuzzy rules (1) if, for each $i \in \{1, 2, ..., n\}$ and for each $A \in \mathcal{F}(X)$, the following inequality holds:

$$\bigwedge_{y \in Y} (B_i(y) \leftrightarrow (f_R^{@}(y))) \geq \bigwedge_{x \in X} (A_i(x) \leftrightarrow A(x))$$

How this definition can be interpreted is, distance between any two fuzzy sets in the codomain of inference mapping is not more than that between the corresponding fuzzy pre-images in the domain. The fuzzy inference depends upon both , the model of fuzzy rules, R and the inference mechanism, @. So, typically, we should talk about the continuity of not just a fuzzy relation but the whole FIS. We just mentioned about the distance between the fuzzy sets. The closeness between the fuzzy sets is computed by the biresiduum operation $\leftrightarrow$. Let us consider a continuous Archimedean t-norm with an additive generator $g : [0, 1] \rightarrow [0, +\infty]$. Then, the biresiduum may be written in the form,

$$a \leftrightarrow b = g^{-1}(|g(a) - g(b)|)$$

where, $g^{-1} : [0, +\infty] \rightarrow [0, 1]$ is the inverse function, where, $g(0) = \infty$ and $g(0) - g(0) = 0$. Hence for a non-empty X, it is possible to define a metric D_g on $\mathcal{F}(X)$ that is generated by g as follows:

$$D_g(A, B) = \bigvee_{x \in X} (|g(A(x)) - g(B(x))|)$$

With this notion, now we can define the continuity of the inference model as:
For the fuzzy rules (1) and a continuous Archimedean t-norm having a continuous additive generator g. A fuzzy relation $R \in \mathcal{F}(X \times Y)$ is a continuous model of the fuzzy rules if and only if

$$D_g(B_i, f_R^{@}(A)) \leq D_g(A_i, A)$$

for each $A \in \mathcal{F}(X)$.
The notion of continuity for Non-FRIs has not been evolved yet.

3.8 Robustness

3.8.1 Robustness

Let X be a classical set, and let $\sim$ be an equivalence relation that is defined on X, i.e., $\sim$ is reflexive, symmetric, and transitive. Immediately, $\sim$ partitions X into equivalence classes. It is well known that an $M \subseteq X$ belongs to this partition if and only if whenever $x \in M$ and $x \sim y$ for some $y \in X$, then $y \in M$. In a sense, the elements of M are indistinguishable and can be represented mathematically as follows:

$$x \in M \text{ and } x \sim y \implies y \in M \ .$$

A fuzzy subset E of the Cartesian product $X \times X$ is called a fuzzy equivalence relation on X if the following properties are satisfied for all $x, y, z \in X$:

- (Reflexivity) $E(x, x) = 1$,
- (Symmetry) $E(x, y) = E(y, x)$,
- (Transitivity) $E(x, z) \geq E(x, y) * E(y, z)$.

A fuzzy set $\mu \in \mathcal{F}(X)$ is called extensional with respect to a fuzzy equivalence relation E on X if

$$\mu(x) * E(x, y) \leq \mu(y), \quad x, y \in X$$

If a fuzzy set μ is not extensional with respect to the considered fuzzy equivalence relation E, the smallest fuzzy set is instead considered, which is extensional with respect to E and contains μ. Let $\mu \in \mathcal{F}(X)$ and let E be a fuzzy equivalence relation on X. The fuzzy set,

$$\hat{\mu}(x) = \bigwedge \{\nu : \mu \leq \nu \textit{ and } \nu \textit{ is an extensional with respect to } E\}$$

is called the extensional hull of μ.
Note : $\mu \leq \nu \implies \mu(x) \leq \nu(x), \ \forall \ x \in X$

Chapter 4

SIRM Connected-Type FIS

Naoyoshi Yubazaki, Jianqiang Yi and Kaoru Hirota proposed a new fuzzy inference model, SIRMs (Single Input Rule Modules) Connected Fuzzy Inference Model [Yubazaki et al.(1997)Yubazaki, Yi, and Hirota] for plural input fuzzy control. For each input item, an importance degree is defined and single input fuzzy rule module is constructed. The importance degrees control the roles of input items in systems. Consider, the rule base given by (1), i.e., in case of conventional fuzzy inference model. We can see that all the input items are put in the antecedent part of each rule. Therefore, the maximum number of fuzzy rules is equal to the number of all combinations of the membership functions among the different input items. Designing fuzzy rules in such fashion is possible when the input items are few but it becomes extremely difficult to establish fuzzy rules when the number of input items increases as every rule is considering all of them. To encounter this problem of conventional fuzzy inference model, SIRMs method was proposed with the following representation of the same rule base as (1) :

SIRM-1: $\{R_j^1 :$ if x_1 is $A_j^1,$ then $y_1 = c_j^1\}_{j=1}^{m_1}$
....
SIRM-i: $\{R_j^i :$ if x_i is $A_j^i,$ then $y_i = c_j^i\}_{j=1}^{m_i}$
....
SIRM-n: $\{R_j^n :$ if x_n is $A_j^n,$ then $y_n = c_j^n\}_{j=1}^{m_n}$

where, SIRM-i denotes the i^{th} single input rule module, and R_j^i is the j^{th} rule in the SIRM-i. x_i is the sole variable in the antecedent part and y_i is the variable in consequent part of the SIRM-i. A_j^i is the fuzzy set in the antecedent, while c_j^i is the crisp output value of the variable y_i in the consequent respectively of rule R_j^i in SIRM-i. Also, i ranges from 1 to n, defining n such SIRMs and j ranging from 1 to m_i indexing m_i rules in SIRM-i. In case , some inputs are contributing more to the model as compared to others, then we can even assign weights to SIRMs corresponding to that input. We do not have this freedom in case of conventional fuzzy inference models.
Inference system performs as follows: Corresponding to each SIRM, output is evaluated by

$$y_i' = \frac{\sum_{i=1}^{m_i} h_j^i c_j^i}{\sum_{i=1}^{n_i} h_j^i}$$

And then the final inference result y' of the entire rule base is given by

$$y' = \sum_{i=1}^{n} w_i y_i'$$

where w_i represents the importance degree of each input item x_i , $(i = 1, 2, ..., n)$.
Note: Here, c_j^is are nothing but constant that depend on input, $c_j^i = f(x_i)$.

4.1 Merits of using SIRMs connected Fuzzy Inference Method

As we have seen, that the whole structure of SIRM method is very simple. It has several other attractions which are stated below :

1. **Sharp reduction in the number of fuzzy rules**
   When in conventional fuzzy models, the total number of rules are given by the number of combinations of the fuzzy sets of all input items, SIRM model has only one variable in the antecedent parts

of its fuzzy rules. Hence, the number of available fuzzy rules in SIRM fuzzy inference model can be computed by taking the sum of the numbers of the membership functions of the input items.
Illustration:

- Given:
  - 3 antecedent fuzzy set domains
  - 5 fuzzy sets on each domain
- Conclusions
  - $5^3 = 125$ input-membership function combinations, i.e., rules to capture the knowledge in some conventional model.
  - $5 + 5 + 5 = 15$ number of rules in SIRM FIS.
  - **A drastic fall in the number of rules**.

2. **Easy design of fuzzy rules**
   As every SIRM takes in only one input variable, its sufficient to design a fuzzy rule exploring the relationship between the current input item and the system performance. Consequently, designing fuzzy rules becomes much easier.

3. **Desired results can be obtained by adjusting the importance degree**
   As said above, SIRM fuzzy inference method gives the freedom of adjusting the importance degrees of the input items. We can give more weight to the input variables corresponding to their contribution in the model, and vice-versa.

4. **Efficiency in Inferencing**
   SIRM fuzzy inference method needs very few fuzzy rules and parameters. This relaxes the demand on memory. Also, the degree of fitness of the input variable becomes the agreement of the antecedent part because of existence of only one input variable. Hence, the time required for inferencing gets reduced considerably.

4.2 Equivalence Between SIRM FIS and Other FIS's

The second of the two approaches suggested to study the properties on SIRM FIS was to check if SIRM FIS is equivalent to some other FIS and see if those properties hold true in the 'other' FIS. This section comprises of some results about equivalence between fuzzy inference systems and they are as follows:

4.2.1 Equivalence between SIRM and SIC FIS

Assume that the weight w_i for Rules-i of the SIRMs method of (12) satisfies the following equation for any x_i :

$$w_i = \frac{\sum_{j_i=1}^{m_1} h_{j_i}^1}{\sum_{i=1}^{n}\sum_{J_i=1}^{m_i} h_{j_i}^i}$$

Then, the inference results by the SIRMs method and SIC method are equal.
From the following equivalent relations between FISs, we can have SIRM FIS equivalent to other conventional FISs as well:

- The inference results by the simplified fuzzy inference method and SIC method are equal when the rules of the simplified fuzzy inference method obtained by the SIC method are used, and the following condition does not depend on the inputs for any $i = 1, 2, ..., n$:

$$\frac{\sum_{j_i=1}^{m_1} h_{j_i}^1}{\sum_{i=1}^{n}\sum_{J_i=1}^{m_i} h_{j_i}^i} = constant$$

  That is, Given a rule base, as given in 4^{th} section, when simplified inference is implemented on the following rule base obtained from SIRM rule base, If $\tilde{x_1}$ is $A^1_{j_1}$, $\tilde{x_2}$ is $A^2_{j_2}$, ... , $\tilde{x_n}$ is $A^n_{j_n}$ then $y = \frac{\sum_{j_i=1}^{m_1} h^1_{j_i}}{\sum_{i=1}^{n}\sum_{J_i=1}^{m_i} h^i_{j_i}} y^1_{j_1} + \ldots + \frac{\sum_{j_i=1}^{m_n} h^n_{j_i}}{\sum_{i=1}^{n}\sum_{J_i=1}^{m_i} h^i_{j_i}} y^n_{j_n}$

then, same outputs will be obtained in both the cases.

Here, $j_i = 1, \ldots, m_i$ and there will be $\prod_{i=1}^{n} m_i$ such rules.

- Let the weight of the fuzzy singleton-type inference method be distributed to the antecedent part of the SIC method. Then, the SIC method can be transformed to the fuzzy singleton-type inference method.
- Let the area of the consequent part of the product sumgravity method be distributed to the antecedent part of the SIC method. Then, the SIC method can be transformed into the productsumgravity method.

Chapter 5

Work Done

Our motive was to study the properties like interpolativity, monotonicity, universal Approximation, etc. on SIRM fuzzy inference model.
In this chapter, we investigate the conditions under which the SIRM FIS is

- Interpolative and
- Monotonic.

The explanation is as follows:

We will show that, given a data set, we can construct an interpolative SIRM rule base. To formulate it, let us fix some notations first:

- n is the number of dimensions/domains.
- $i : \{1, \ldots, n\}$
- $\mathcal{U}_i$ - the antecedent fuzzy sets domains,
- $\tilde{x_i}$ - the linguistic variables of the input fuzzy sets.
- $\mathcal{V}$ - the sole codomain for the rule base.
- k_i - the number of fuzzy sets on the domain $\mathcal{U}_i$
- $j : \{1, \ldots, k_i\}$
- A_i^j - the input fuzzy sets of the j^{th} rule in the i^{th} module
- c_i^j - the constants in the consequents.

Theorem 5.0.1. *Let*

- $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \cdots \times \mathcal{U}_n$ *be the n-dimensional input domain,*
- $\mathcal{V}$ *be the output domain,*
- $\mathcal{D} = \{(\bar{\alpha}_k\ ,\ \beta_k) \in \mathcal{U} \times \mathcal{V}\}_{k=1}^{\ell}$ *be the set of data points given about the system,*
- *where* $\bar{\alpha}_k = (\alpha_{k1}, \ldots, \alpha_{kn}) \in \mathcal{U}$.

Then one can construct a rule base of SIRM such that

- $\{A_i^k\}_{k=1}^{\ell}$ *are the normal antecedent fuzzy sets,*
- α_{k_i} *are the points of normality of* A_i^k,
- $\{A_i^k\}_{k=1}^{\ell}$ *form a Ruspini Partition on corresponding* $\mathcal{U}_i$, *and*

The SIRM Rule base whose i^{th} module is given as,

$$\mathbb{M}_i : \begin{cases} \text{If } \widetilde{x}_i \text{ is } A_i^1 \text{ Then } y \text{ is } c_i^1 \\ \vdots \\ \text{If } \widetilde{x}_i \text{ is } A_i^l \text{ Then } y = c_i^{k_l} \end{cases}$$

where, c_i^k are the solutions of the following system:

$$x_1^1 + x_2^1 + \cdots + x_n^1 = \beta_1$$
$$\vdots$$
$$x_1^\ell + x_2^\ell + \cdots + x_n^\ell = \beta_\ell$$

will be interpolative when the SIRM inference is employed.

Proof. Let us first understand the theorem in two-dimensional case with two fuzzy sets in each domain.

Further suppose that (α_1,β_1) and (α_2,β_2) be the two dimensional inputs to the FIS and γ_1 and γ_2 be the corresponding outputs. Then, from the above theorem we can construct an interpolative SIRM rulebase, as given below:

$$\mathbb{M}_1 : \begin{cases} \text{If } \widetilde{x_1} \text{ is } A_1^1 \text{ Then y} = c_1^1. \\ \text{If } \widetilde{x_1} \text{ is } A_1^2 \text{ Then y} = c_1^2. \end{cases}$$

$$\mathbb{M}_2 : \begin{cases} \text{If } \tilde{x}_2 \text{ is } A_2^1 \text{ Then y} = c_2^1. \\ \text{If } \tilde{x}_2 \text{ is } A_2^2 \text{ Then y} = c_2^2. \end{cases}$$

While constructing the rule base, we are given the inputs and the corresponding outputs. We have to design the fuzzy sets and find the constants in the consequent part , so that the inference system becomes interpolative.
According to the inference mechanism in case of SIRM FIS,

$$y_1' = \frac{A_1^1(\alpha_1)c_1^1 + A_1^2(\alpha_1)c_1^2}{A_1^1(\alpha_1) + A_1^2(\alpha_1)}$$

$$y_2' = \frac{A_2^1(\beta_1)c_2^1 + A_2^2(\beta_1)c_2^2}{A_2^1(\beta_1) + A_2^2(\beta_1)}$$

Hence, the final output corresponding to (α_1, β_1) will be evaluated as,

$$\gamma_1 = \frac{A_1^1(\alpha_1)c_1^1 + A_1^2(\alpha_1)c_1^2}{A_1^1(\alpha_1) + A_1^2(\alpha_1)} + \frac{A_2^1(\beta_1)c_2^1 + A_2^2(\beta_1)c_2^2}{A_2^1(\beta_1) + A_2^2(\beta_1)}$$

For the sake of simplicity, let us assume that there exists Ruspini partition in the antecedent domains, so that,

$$A_1^1(\alpha_1) + A_1^2(\alpha_1) = 1$$

and,

$$A_2^1(\beta_1) + A_2^2(\beta_1) = 1.$$

Therefore,

$$\gamma_1 = A_1^1(\alpha_1)c_1^1 + A_1^2(\alpha_1)c_1^2 + A_2^1(\beta_1)c_2^1 + A_2^2(\beta_1)c_2^2$$

Let α_1 and β_1 be the points of normality of A_1^1 and A_2^1 respectively. Since we are assuming that its fuzzy psrtition in the input domains, so $A_1^2(\alpha_1) = 0$ and $A_2^1(\beta_1) = 0$. Hence, and giving equal weights to both the modules, $\frac{1}{2}$, the above expression becomes,

$$2\gamma_1 = c_1^1 + c_2^1 \tag{5.1}$$

Similarly, in case of second output, (α_2, β_2), we have,

$$2\gamma_2 = c_1^2 \;+\; c_2^2 \tag{5.2}$$

The system of equations, (12) and (13), can be represented as (in matrix form),

$$Mc = G$$

where,

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$x = \begin{bmatrix} x_1^1 \\ x_1^2 \\ x_2^1 \\ x_2^2 \end{bmatrix}$$

$$G = \begin{bmatrix} 2\gamma_1 \\ 2\gamma_2 \end{bmatrix}$$

and has 4 unknowns in two equations with rank $[M : G] \;=\; 2 \leq 4$ and hence infinitely many solutions. One such solution is,

$$c_1^1 \;=\; c_2^1 \;=\; \gamma_1$$

$$c_1^2 \;=\; c_2^2 \;=\; \gamma_2$$

Hence, we conclude that given a data set, we can have an interpolative SIRM rule base with constants in the consequent part of rules as the solutions of the system (12) and (13).

In the similar way, we can generalise the above result to any dimension n.

In n dimentional case, with the given hypothesis, we can have an interpolative SIRM rule base with constants in its consequents satisfying the system of equations represented as : $Mc \;=\; G$, where,

$$M = \begin{bmatrix} I_l & I_l & ... & I_l \end{bmatrix}$$

where I_l is the identity matrix of order l and M will be a $l \times n$ matrix having n blocks of order l.

So, in this case also the system will have infinitely many solutions with one solution is when all the constants relating to the k-th rule in every module is equal, i.e.,

$$c_i^k = \beta_k \;,\; \forall i = \{1, \ldots, n\}$$

Remark 5.0.2. *SIRM FIS is very simple to handle as it breaks down the entire rule base into modules containing SISO rules, which are easy to process. We saw that SIRM will be interpolative, however large n we take.*

Remark 5.0.3. *A thing to be noted is that the number of fuzzy sets forming a ruspini partition on the domain should be equal to the number of the points in the input data. The reason for this is that we want to construct a rulebase according to the given data set, for which we are trying to find the constants* c_i^j*s with the help of the given data set. And we want the input vector,* $\bar{\alpha}_k = (\alpha_{k1}, \ldots, \alpha_{kn})$ *such that its* i^{th} *component* α_{ki} *is the point of normality of some fuzzy set in the domain* $\mathcal{U}_i\ \forall\ i = 1, ..., n$. *So we want atleast l (thats the number of points in the dataset) fuzzy sets in each domain and if we will take more than l fuzzy sets, then the system Mx=G will always be inconsistent.*

□

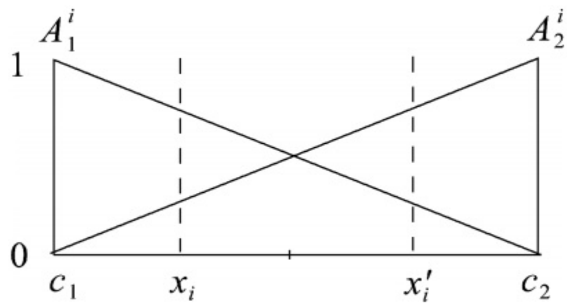

Figure 5.1:

5.1 Monotonicity of SIRM FIS

Monotonicity of SIRM FIS has already been talked about by Seki etal in their works [Hirosato Seki and Mizumoto(2010)] and [Seki and Tay(2012)]. We studied their work critically and observed the following:

- In [Hirosato Seki and Mizumoto(2011)], equivalence between TSK and Functional-Type SIRM FIS (where consequents of the rules are the functions of the input varialble) has been shown and later in their work, [Hirosato Seki and Mizumoto(2010)], they proved the monotonicity of Functional-type SIRM FIS. But there, they have considered the fuzzy sets as given in the figure (5.1) only.

  They say that, given two fuzzy sets A_1^i and A_2^i (where i denotes the i^{th} domain) forming fuzzy partition, the inference result obtained by the functional-type SIRMs method is monotonically increasing, if the fuzzy sets form a fuzzy partition, as shown in the above figure, and if the consequent parts are monotonically increasing.
  But, they have not considered the cases wherein both x_i and x_i' lie on either the left of the intersecting point or on the right of the intersecting point, which consequently make the proof incomplete.

- Seki and Tay in their work, [Seki and Tay(2012)], have again tried to show that SIRM FIS is monotonic, but this they are proving it independently for SIRM FIS, without comparing it with any other FIS. But in this work as well, they are missing the above mentioned cases, and proving the monotoncity of SIRM FIS for a particular case.

In our this work we are giving an aliter to their proofs, perhaps, overcoming the drawbacks of their proofs.

Theorem 5.1.1. *Given monotonic inputs to a monotonic rule base, the corresponding outputs obtained by inferring from it using SIRM FIS will also be monotonic.*

Proof. From 4.2 section, we have results on equivalence between SIRM FIS and SIC FIS and between SIC and simplified FIS, Product-sum gravity method and Fuzzy singleton FIS.
We also know that TSK model is monotonic from the work by Seki etal. in [Seki and Tay(2012)]. We also know that simplified fuzzy inference system is a particular case of TSK FIS (In this case, the consequents are constant functions rather than other general functions).
So, we have that SIRM FIS is also monotonic. □

Using earlier established results, it was very easy to observe the monotonicity of SIRM FIS. We also have a remark on this which is as follows:

Remark 5.1.2. *In all of the above results on equivalence, proved in [Hirosato Seki and Mizumoto(2011)], seki etal. have assumed the following condition:*

$$\frac{\sum_{j_i=1}^{m_1} h_{j_i}^1}{\sum_{i=1}^{n}\sum_{J_i=1}^{m_i} h_{j_i}^i} = constant$$

and they want this condition to be satisfied independent of the input, which is a very tight condition, and practically very hard to satisfy. But we see that, if we assume that there is fuzzy partition in the antecedent fuzzy sets, which almost every application requires, then this constant is nothing but $\frac{1}{n}$*, where* n *is the number of dimensions. And hence, SIRM FIS will always be equivalent to simplified fuzzy inference method in case of ruspini partition in the antecedent fuzzy sets and since TSK FIS is monotonic (from [Seki and Tay(2012)]), we have SIRM FIS also monotonic.*

Chapter 6

Summary

The whole work is centered around fuzzy inference models. It gives a brief about all the fuzzy inference models, that are used in almost all the applications (there are about 6 such models). It explains the newly introduced SIRM FIS in detail and the equivalence between it and the other FISs. The motivation for this work came from the fact that SIRM FIS has not been studied much and because of its several advantages.
This work also explains about the basic desirable properties that every FIS is expected to have and they are, Interpolativity, Continuity, Monotonicicty, Robustness and Universal Approximation. Then since very few results are available regarding SIRM FIS having those properties, we tried to investigate if they hold true in case of SIRM FIS and we concluded that this inference model is **interpolative** as well as **monotonic**. We stated two approaches to prove that. While the first approch helped us to prove former, the latter got proved using the second.
However, we are yet to see other properties for SIRM FIS.

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