Macrospin in external magnetic field: Entropy production and fluctuation theorems

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We consider stochastic rotational dynamics of a single magnetic domain in an external magnetic field, at constant temperature. Starting from the appropriate Langevin equation of motion, we calculate entropy production along stochastic trajectories to obtain fluctuation theorems, by presenting several possibilities of choosing conjugate trajectories. One of these choices gives entropy production in the reservoir that is consistent with prediction from Fokker-Planck equation. We further show the relation between heat absorbed and entropy production in the reservoir, using stochastic energy balance. For a time-independent field, the magnetization obeys Boltzmann distribution, however, also supports an azimuthal current making the dynamics inherently non-equilibrium. We use numerical simulations to obtain distribution functions for entropy production along trajectories which show good agreement with the detailed fluctuation theorem.

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I. INTRODUCTION

With miniaturization of memory devices like the magnetic read head and random access memory, thermal fluctuations start to play non-trivial role in their performance, e.g., by activating magnetization reversal of ferromagnetic clusters [1]. The impact of thermal noise is stronger in smaller devices [2], with the intensity inversely proportional to system size. The thermally induced magnetization fluctuations will act as a fundamental limit to the performance of submicron magnetoresistive devices. Thus it becomes crucial to understand the impact of thermal fluctuations, in order to reliably use small magnetic devices [3–8].

Recent theoretical developments allows us to describe equilibrium and non-equilibrium properties of small systems under strong thermal fluctuations in terms of stochastic thermodynamics [9–11]. Defining stochastic counterparts of thermodynamic observables, e.g., stochastic energy, work, heat, and entropy characterizing individual trajectories in phase space relied on quantities, statistical averages of which denote the thermodynamic variables. Several equalities involving these quantities have been derived in last two decades [12–19]. It was identified that negative entropy producing trajectories are a possibility, but exponentially suppressed with respect to positive entropy producing trajectories and the corresponding equality is known as detailed fluctuation theorem [20–27]. A related integral fluctuation theorem, and the Jarzynski equality that relates equilibrium free energy difference to non-equilibrium work done was derived [12, 23]. Many of these theorems have been verified against experiments on colloids and granular matter [28–31], and successfully used to obtain free energy landscape of bio-polymers like RNA [32, 33]. Stochastic thermodynamics has recently been extended to describe active Brownian particles that derives their motion using internal energy source or ambient fuel [34–38].

In a previous study, work distribution functions of a single spin obeying Glauber dynamics was obtained under various protocols of changing magnetic field [39]. The focus of current paper is to study stochastic thermodynamics of the simplest miniaturized magnetic system, a single domain magnetic particle or macrospin, undergoing Langevin dynamics in presence of an external magnetic field [40, 41]. A sufficiently small ferromagnetic particle may have a single domain where all the spins are aligned along a specific direction via the exchange interaction. This super-paramagnetic particle at temperatures below the blocking temperature, behaves like a single spin entity with magnetization $\vec{m}$. Due to the small system size the dynamics of such a *spin* is strongly influenced by the thermal noise $\vec{h}(t)$.

In the following section we present the Langevin dynamics of such a macrospin. Using Fokker-Planck equation we obtain the time-evolution of stochastic entropy to show that upon averaging, the total entropy production becomes positive as required by the second law of thermodynamics. Starting from the stochastic equation of motion, we then derive expressions for entropy production in the system and in the heat bath along stochastic trajectories. This allows us to derive fluctuation theorems involving entropy production. We present several possibilities of choosing the conjugate trajectories, and discuss their implications. In presence of a constant magnetic field, the system shows a curious dichotomy. While the axial component of probability current vanishes and the probability distribution obeys Boltzmann statistics, the azimuthal current remains finite [3], which in turn supports a positive entropy production. We obtain the

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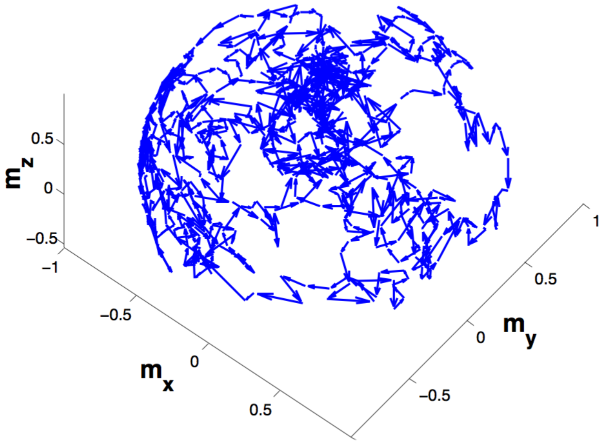

FIG. 1: (Color online) A typical trajectory of magnetization $\vec{m}$ in presence of a magnetic field $\vec{H} = H\hat{z}$ with $H = 1$ and a Langevin heat bath at temperature $k_B T = 1$, obtained from numerical simulations. The arrow heads denote the direction of motion.

probability distribution of entropy production, using numerical simulations of the model. This distribution function obeys the detailed fluctuation theorem.

II. MODEL

The stochastic dynamics of a magnetic particle with magnetization $\vec{m}$ under an external field $\vec{H}$ and in contact with a heat-bath is described by the Landau-Lifshitz-Gilbert (LLG) equation

$$\frac{d\vec{m}}{dt} = \gamma\, \vec{m} \times \left[\vec{H} + \vec{h}(t) - \eta \frac{d\vec{m}}{dt}\right], \tag{1}$$

where γ denotes the gyromagnetic ratio. The Langevin heat bath is characterized by the viscous dissipation $-\eta\, d\vec{m}/dt$, and the Gaussian white noise $\vec{h}(t)$ the components of which obey $\langle h_i(t)\rangle = 0$, $\langle h_i(t) h_j(t')\rangle = 2D_0 \delta_{ij}\delta(t - t')$ with $D_0 = \eta k_B T/V$, where T is the temperature, and k_B Boltzmann constant, and V denotes the volume of the magnetic particle [3]. Here components i, j denote the cartesian x, y, z coordinates. We assume that the statistical properties of $\vec{h}(t)$ are isotropic. In the above equation $\vec{H}$ denotes the conservative field $\vec{H} = -\partial U/\partial \vec{m}$ where $U = -\vec{m}.\vec{H}$ is the Gibb's free energy per unit volume. The macrospin undergoes a relaxation dynamics in the Langevin heat bath, settling into an average unidirectional precession around the field $\vec{H}$.

We assume the amplitude of the magnetization $m = |\vec{m}|$ remains unchanged. The angular dynamics of the instantaneous orientation of magnetization (θ, ϕ) on a unit sphere determined by LLG equation can be written as

$$\begin{aligned} \frac{\partial\theta}{\partial t} &= h'\, m(H_\theta + h_\theta) - g' \frac{m}{\sin\theta}(H_\phi + h_\phi) \\ \frac{\partial\phi}{\partial t} &= g' \frac{m}{\sin\theta}(H_\theta + h_\theta) + h' \frac{m}{\sin^2\theta}(H_\phi + h_\phi) \end{aligned} \tag{2}$$

with

$$g' = \frac{1/\gamma}{m[(1/\gamma^2) + \eta^2 m^2]},\; h' = \frac{\eta}{(1/\gamma^2) + \eta^2 m^2},$$

where $H_\theta = -(1/m)\partial_\theta U$, $H_\phi = -(1/m\sin\theta)\partial_\phi U$ are the components of external magnetic field such that $\vec{H} = \hat{\theta} H_\theta + \hat{\phi} H_\phi$, and $h_\theta = h_x \cos\theta\cos\phi + h_y \cos\theta \sin\phi - h_z \sin\theta$, $h_\phi = -h_x \sin\theta\sin\phi + h_y \sin\theta\cos\phi$ are the components of stochastic fields. For generality, here we assumed $U(\theta, \phi)$. The statistical properties of cartesian coordinates of Gaussian white noise $\vec{h}$ were given above. It should be noted that the radial component of the stochastic field $h_r = h_x \sin\theta\cos\phi + h_y \sin\theta\sin\phi + h_z\cos\theta$ does not appear in the angular motion of magnetization.

A. Fokker-Planck equation and entropy production

As the amplitude of magnetization $|\vec{m}|$ is conserved during the dynamics, its instantaneous orientation (θ, ϕ) can be represented by a point on the unit sphere. A statistical ensemble of such points on the surface of a sphere can be described by surface density $W(\theta, \phi, t)$. The corresponding Fokker-Planck equation is expressed as [3]

$$\frac{\partial W}{\partial t} = -\vec{\nabla}_\Omega.\vec{J}_\Omega, \quad \vec{J}_\Omega = \hat{\theta} J_\theta + \hat{\phi} J_\phi \tag{3}$$

where the two-dimensional divergence on the surface of the unit sphere $\vec{\nabla}_\Omega.\vec{J}_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta J_\theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} J_\phi$, Ω denotes a solid angle. The two components of dissipative current are expressed as [3]

$$\begin{aligned} J_\theta &= m[h' H_\theta - g' H_\phi] W + k' \partial_\theta W \\ J_\phi &= m[g' H_\theta + h' H_\phi] W - \frac{k'}{\sin\theta}\partial_\phi W. \end{aligned} \tag{4}$$

In the above relations, h' and g' play the role of mobility, and $k' = D_0 m^2 (h'^2 + g'^2)$ plays the role of diffusivity for angular dynamics.

The non-equilibrium Gibbs entropy $S = -k_B \int d\Omega\, W \ln W(\theta, \phi, t)$ where $d\Omega = \sin\theta\, d\theta\, d\phi$ denotes solid angle, suggests a definition of time dependent stochastic entropy of the system $s(t) = -k_B \ln W(\theta, \phi, t)$ where the total entropy $S = \langle s\rangle$ is the ensemble average of the stochastic entropy [24]. Thus the rate of change in entropy $\dot{s} \equiv ds/dt$ is

$$\begin{aligned} \frac{\dot{s}}{k_B} &= -\frac{\partial_t W}{W} - \frac{\partial_\theta W}{W}\dot{\theta} - \frac{\partial_\phi W}{W}\dot{\phi} \\ &= -\frac{\partial_t W}{W} + \frac{J_\theta \dot{\theta} + J_\phi \sin\theta\, \dot{\phi}}{k' W} - \frac{\dot{s}_r}{k_B} \end{aligned} \tag{5}$$

where

$$\frac{\dot{s}_r}{k_B} = \frac{m}{k'}\left[h'(H_\theta\dot{\theta} + H_\phi \sin\theta\dot{\phi}) + g'(H_\theta \sin\theta\dot{\phi} - H_\phi\dot{\theta})\right]$$
$$= \frac{1}{mD_0}\left[c_1(H_\theta\dot{\theta} + H_\phi \sin\theta\dot{\phi}) + c_2(H_\theta \sin\theta\dot{\phi} - H_\phi\dot{\theta})\right]. \tag{6}$$

The first step in Eq.(5) identifies the explicit and implicit time dependences. The second step is obtained by using Eq.(4) to replace $\partial_\theta W$ and $\partial_\phi W$. In obtaining the second step in Eq.(6) we used $k' = D_0 m^2(h'^2 + g'^2)$, and the relations $c_1 = h'/(h'^2 + g'^2)$, $c_2 = g'/(h'^2 + g'^2)$. Note that the expression $\dot{s}_r$ consists of terms with dimensions of torque times angular velocity, similar to dissipated work that one obtains from usual Langevin dynamics of particles. Interpreting s_r as the entropy production in the reservoir, the rate of change in total stochastic entropy,

$$\frac{\dot{s}_t}{k_B} = \frac{1}{k_B}(\dot{s} + \dot{s}_r) = -\frac{\partial_t W}{W} + \frac{J_\theta\dot{\theta} + J_\phi \sin\theta\,\dot{\phi}}{k'W}. \tag{7}$$

The total entropy production $\dot{S}_t = \langle \dot{s}_t \rangle$ should be ≥ 0 by second law of thermodynamics. The averaging here involves two steps, one over trajectories, and the other over all solid angles Ω. The trajectory average of the components of angular velocity $\langle \dot{\theta}|\,\theta,\phi,t\rangle = J_\theta/W$ and $\langle \sin\theta\,\dot{\phi}|\,\theta,\phi,t\rangle = J_\phi/W$ [24]. The conservation of probability leads to $\int d\Omega\,\partial_t W = 0$. As a result,

$$\dot{S}_t = \langle \dot{s}_t \rangle = \int d\Omega \frac{J_\theta^2 + J_\phi^2}{k'W} \geq 0, \tag{8}$$

where the equality requires both J_θ and J_ϕ to be zero, a result expected from the second law of thermodynamics.

B. Fluctuation theorems

Now we proceed to derive entropy production along stochastic trajectories. Physically, entropy production characterizes the irreversibility of a path. Consider the time evolution of a macrospin from $t = 0$ to τ_0 through a path $X = [\theta(t), \phi(t), \vec{H}(t)]$, assuming for the moment a time-dependent protocol of controlling $\vec{H}$. The motion along this trajectory is governed by stochastic forces via the coupling with Langevin heat bath. Let us divide the whole trajectory into $i = 1,\ldots,N$ segments of time-interval δt such that $N\delta t = \tau_0$. The transition probability $p_i^+(\theta',\phi',t+\delta t|\theta,\phi,t)$ on i-th infinitesimal segment is governed by the Gaussian random noise at i-th instant $P(\vec{h}_i) = (\delta t/4\pi D_0)^{1/2}\exp(-\delta t\,\vec{h}_i^2/4D_0)$ where $\vec{h}_i^2 = h_x^2 + h_y^2 + h_z^2$ calculated at that instant. For ease of expression, let us denote the time-evolution of Eq.(2) using the form $\dot{\theta} = \Theta(\theta,\phi,\vec{H})$ and $\dot{\phi} = \Phi(\theta,\phi,\vec{H})$ where $\dot{\theta} = \partial_t\theta$ and $\dot{\phi} = \partial_t\phi$. Thus the transition probability on i-th segment $p_i^+ = \mathcal{J}\langle\delta(\dot{\theta}-\Theta)\delta(\dot{\phi}-\Phi)\rangle = \mathcal{J}\int d\vec{h}_i P(\vec{h}_i)\delta(\dot{\theta}-\Theta)\delta(\dot{\phi}-\Phi)$. The integration over the Dirac-δ functions are performed after writing $\vec{h}_i$ in terms of its spherical polar components with the constraint that the radial part of the stochastic fluctuation is set to zero. This is required as the magnitude of magnetization is assumed to be constant. The quantity $\mathcal{J}$ denotes the Jacobian of transformation $\mathcal{J} = \det[\partial(h_\theta,h_\phi)/\partial(\theta,\phi)]$. This can be evaluated easily within the Stratonovich discretization of Eq.2. The probability of a full trajectory in time-forward evolution is $\mathcal{P}_+ = \prod_{i=1}^N p_i^+$.

It is possible to choose conjugate dynamics and trajectories in several ways [11, 42]. In choosing the time-reversed trajectory, let us first start from the angular dynamics Eq.(2). In order to characterize the irreversibility of time-forward path, we initiate the time-reversed trajectory from the micro-state characterized by the final value of control field $\vec{H}$ reached at the end of time-forward path, and trace it back. Thus we consider the time-reversed trajectory $X^\dagger = [\theta(\tau_0-t), \phi(\tau_0-t), \vec{H}(\tau_0-t)]$, considering θ, ϕ and $\vec{H}$ as even functions under time reversal [43]. The time-reversed trajectory includes reversal of sign of all the odd variables $\dot{\theta}$ and $\dot{\phi}$. The probability of time-reversed path may again be divided into N segments such that the probability of full trajectory $\mathcal{P}_- = \prod_{i=1}^N p_i^-$ where $p_i^- = \mathcal{J}\int d\vec{h}_i P(\vec{h}_i)\,\delta(\dot{\theta}+\Theta(\tau_0-t)\,)\,\delta(\dot{\phi}+\Phi(\tau_0-t)\,)$. The Jacobian of transformation being independent under time-reversal drops out from the ratio p_i^+/p_i^-.

Thus the ratio of these two probabilities of the forward and reverse trajectories is given by $\frac{\mathcal{P}_+}{\mathcal{P}_-} = \exp(\Delta s_r/k_B)$ where (see Appendix A)

$$\frac{\Delta s_r}{k_B} = \frac{1}{mD_0}\int_0^{\tau_0} dt\left[H_\theta(c_1\dot{\theta} + c_2\sin\theta\,\dot{\phi}) + H_\phi\left(c_1\sin\theta\,\dot{\phi} - c_2\dot{\theta}\right)\right], \tag{9}$$

with $c_1 = h'/(h'^2 + g'^2)$, $c_2 = g'/(h'^2 + g'^2)$. Let us assume that the trajectories considered above describe evolution from initial steady state described by a distribution $W_i(\theta_i,\phi_i,\vec{H}_i)$ to a final state $W_f(\theta_f,\phi_f,\vec{H}_f)$.

We consider the quantity $R = W_i\mathcal{P}_+/W_f\mathcal{P}_-$, which denotes the ratio of probabilities of time-forward and time-reversed evolution along all such trajectories. In a situation obeying time-reversal symmetry $R = 1$, denoting equilibrium and hence produces no entropy. Deviation of R from unity leads to entropy production.

The change in stochastic entropy of the system $\Delta s = s_f - s_i = k_B \ln(W_i/W_f)$. This is a state function and depends on the exact initial and final micro-states. On the other hand, the change in entropy in the reservoir depends on the trajectory and is given by $\Delta s_r = k_B \ln(\mathcal{P}_+/\mathcal{P}_-)$ [44]. The total entropy change $\Delta s_t = k_B \ln R = k_B \ln(W_i\mathcal{P}_+/W_f\mathcal{P}_-) = \Delta s + \Delta s_r$. This relation readily leads to the integral fluctuation theorem [11], $\langle e^{-\Delta s_t/k_B}\rangle = 1$. Note that in deriving this relation, $\sum_X \equiv \sum_X^\dagger$ is used, meaning that the transformation from time-forward path X to time-reversed path $X^\dagger$ has a Jacobian of unity [43]. Further, in a steady state the total entropy change along a time-forward path Δs_f is equal and opposite to that along the time-reversed path, $\Delta s_f(X) = -\Delta s_r(X^\dagger)$. Using this, one obtains the following detailed fluctuation theorem [16, 23]

$$\rho(\Delta s_t) \;=\; e^{\Delta s_t/k_B}\rho(-\Delta s_t). \tag{10}$$

We present three more choices of conjugate trajectories and their consequences. Instead of considering Eq.(2) as the one governing dynamics, if we go one step back and consider the original macrospin dynamics Eq.(1), we see that both $\vec{m}$ and $\vec{H}$ are odd functions under time reversal. Denoting the path probability of such an asymmetric reverse trajectory $X^\dagger = [-\vec{m}(\tau_0-t), -\vec{H}(\tau_0-t)]$ with $\mathcal{P}_-^{(1)}$, the ratio $\mathcal{P}_+/\mathcal{P}_-^{(1)} = \exp(\Delta s_r^{(1)}/k_B)$ where

$$\frac{\Delta s_r^{(1)}}{k_B} = \frac{c_1}{mD_0}\int_0^{\tau_0} dt\left[H_\theta\dot{\theta} + H_\phi \sin\theta\,\dot{\phi}\right]. \tag{11}$$

Similarly as above, it can be shown that $\Delta s_t^{(1)} = \Delta s + \Delta s_r^{(1)}$ obeys the integral fluctuation theorem. As the external driving $\vec{H}$ in the present case is not symmetric under time reversal, the detailed fluctuation theorem will have the form $\rho(\Delta s_t) = e^{\Delta s_t/k_B}\rho^\dagger(-\Delta s_t)$ where $\rho^\dagger$ denotes the probability calculated along the conjugate trajectory.

For the choice of conjugate trajectory in which $\vec{H}$ alone changes sign such that $X^\dagger = [\vec{m}(\tau_0-t), -\vec{H}(\tau_0-t)]$, denoting the path probability of conjugate trajectory $\mathcal{P}_-^{(2)}$, one obtains the ratio $\mathcal{P}_+/\mathcal{P}_-^{(2)} = \exp(\Delta s_r^{(2)}/k_B)$ where

$$\frac{\Delta s_r^{(2)}}{k_B} = \frac{c_2}{mD_0}\int_0^{\tau_0} dt\left[H_\theta \sin\theta\,\dot{\phi} - H_\phi\dot{\theta}\right]. \tag{12}$$

Again, $\Delta s_t^{(2)} = \Delta s + \Delta s_r^{(2)}$ obeys the integral and the appropriate detailed fluctuation theorems.

The third alternative is to consider conjugate trajectories in which $\vec{m}$ alone changes sign, i.e., $X^\dagger = [-\vec{m}(\tau_0 - t), \vec{H}(\tau_0-t)]$. Denoting the probability of conjugate trajectory $\mathcal{P}_-^{(3)}$, one obtains $\mathcal{P}_+/\mathcal{P}_-^{(3)} = 1$, i.e., the corresponding stochastic entropy for trajectory $\Delta s_r^{(3)} = 0$. The resultant total entropy $\Delta s_t^{(3)} = \Delta s$ also obeys the integral and detailed fluctuation theorems.

Among the various possible choices of entropy associated to stochastic trajectories, the definition of Δs_r in Eq.(9) obtained from using the time-reversed trajectory in (θ,ϕ)-space, directly leads to the expression $\dot{s}_r$ in Eq.6 obtained from Fokker-Planck equation. Note that the derivation of $\dot{s}_r$ in Eq.6 does not depend on any particular choice of conjugate trajectory, rather only on the dynamics. Also, $\Delta s_r = \Delta s_r^{(1)} + \Delta s_r^{(2)}$. In the following, we further explore meaning of entropy production in this system.

C. Detailed balance condition

Let us now discuss the detailed balance condition, and its implications. In presence of an uniaxial external field, the potential energy per unit volume $U(\theta) = -H\,m\cos\theta$, i.e., $H_\phi = 0$ as $\partial_\phi U = 0$. Assuming the same uniaxial symmetry in probability distribution $W(\theta,t)$ as in potential energy $U(\theta)$, $\partial_\phi W = 0$, leading to a J_ϕ that is independent of ϕ such that $\partial_\phi J_\phi = 0$ (see Eq.4). Thus the Fokker-Planck equation reduces to

$$\frac{\partial W(\theta)}{\partial t} = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, J_\theta), \tag{13}$$

an equation independent of the variable ϕ. Therefore, the equilibrium detailed balance in the θ-space requires vanishing of the dissipative current $J_\theta = 0$ leading to the canonical Boltzmann distribution $W = W_0\exp[-U(\theta)/k_BT]$. Note that this *equilibrium* allows for the presence of a divergenceless current in the azimuthal direction $J_\phi = -g'(\partial_\theta U)W_0\exp[-U(\theta)/k_BT]$ [3]. This is due to the breakdown of time-reversal symmetry by external magnetic field leading to an average unidirectional precessional motion along ϕ. In this state, though $J_\theta = 0$, $J_\phi \neq 0$. The loss of time-reversal symmetry leads to *non-equilibrium* entropy production $\dot{S}_t > 0$ [Eq.(8)], although maintaining the *equilibrium* distribution in $W(\theta)$. In Ref.[45] an illustrative example, similar to the current situation was considered, in which a two-dimensional particle in a harmonic trap obeying Boltzmann distribution undergoes rotational motion due to externally applied torque.

Let us now obtain the simplified expression of entropy production in reservoir, in presence of the uniaxial symmetry such that $H_\phi = 0$. The expressions for h' and g' lead to the identities $c_1 = \eta m^2$, $c_2 = m/\gamma$. Using these relations and $D_0 = \eta k_BT/V$ in Eq.(9) one finds

$$\frac{\Delta s_r}{k_B} \;=\; \frac{mV}{k_BT}\int_0^{\tau} dt\left[H_\theta\dot{\theta} + \frac{1}{m\eta\gamma}H_\theta\sin\theta\,\dot{\phi}\right]. \tag{14}$$

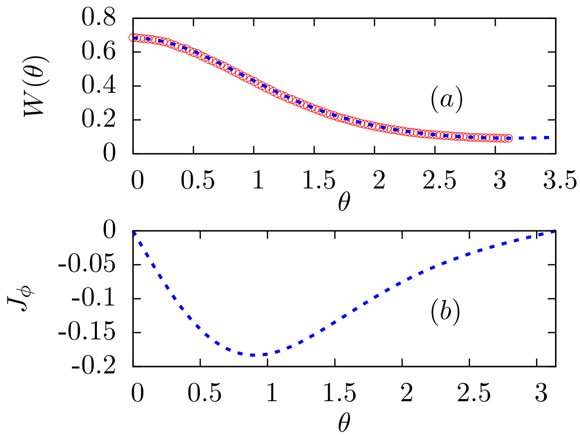

FIG. 2: (Color online) (a) The data points show equilibrium probability distribution $W(\theta)$ obtained from numerical integration of Eq.(2) of magnetization amplitude $m = 1$ at $H\hat{z}$ with $H = 1$, and $k_B T = 1$. The dashed line denotes the analytic function $W_0 \exp(\cos\theta)$ describing the equilibrium distribution. (b) Azimuthal current J_ϕ as a function of polar angle θ.

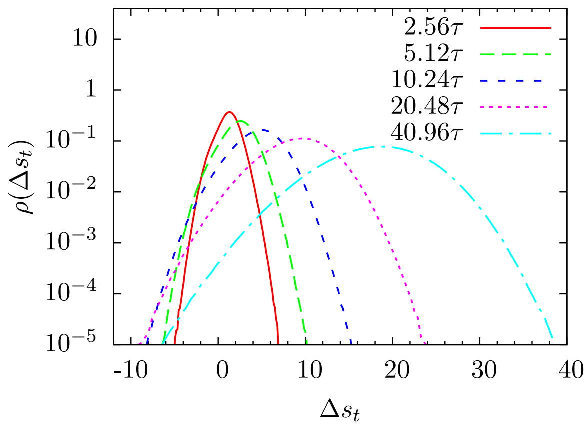

FIG. 3: (Color online) Probability distribution of total entropy production $\rho(\Delta s_t)$ calculated in the presence of a constant external field $H\hat{z}$ with $H = 1$. The calculations are performed after collecting data over $\tau_0 = 2.56, 5.12, 10.24, 20.48, 40.96\tau$.

For a time-independent external field $\vec{H} = H\hat{z}$, the initial and final states are denoted by the *equilibrium* probability distribution $W_{i,f} = W_0 \exp[-U(\theta_{i,f})/k_B T]$, and the change in system entropy

$$\frac{\Delta s}{k_B} = [U(\theta_f) - U(\theta_i)] = -Hm[\cos\theta_f - \cos\theta_i]. \quad (15)$$

D. Stochastic energy balance

To calculate the stochastic energy gain $dU/dt = -\vec{H}.d\vec{m}/dt - \vec{m}.d\vec{H}/dt$, we need a closed form expression for $d\vec{m}/dt$. By taking a cross product of $\vec{m}$ with both sides of the LLG equation, Eq.(1), one obtains

$$\vec{m} \times \frac{d\vec{m}}{dt} = \gamma \vec{m} \times (\vec{m} \times \vec{H}_e) + \eta\gamma m^2 \frac{d\vec{m}}{dt},$$

where $\vec{H}_e = \vec{H} + \vec{h}(t)$, and we used $\vec{m}.d\vec{m}/dt = 0$ as m^2 is constant. Using this relation back in the LLG equation, we get

$$\frac{d\vec{m}}{dt} = mg'\, \vec{m} \times \vec{H}_e - h' \vec{m} \times (\vec{m} \times \vec{H}_e). \quad (16)$$

Then it is straightforward to show that the rate of change in energy $\dot{U} \equiv \frac{dU}{dt}$ is given by

$$\dot{U} = -mg'\, \vec{H}.(\vec{m} \times \vec{h}) + h'\, \vec{H}_e.[\vec{m} \times (\vec{m} \times \vec{H})] - \vec{m}.\dot{\vec{H}}.$$

Here, rate of work done by the magnetic field is $\dot{W} = -\vec{m}.\dot{\vec{H}}$, and $\dot{U} = \dot{q} + \dot{W}$ where q denotes the heat absorbed from the reservoir per unit volume. This is a stochastic version of the first law of thermodynamics. Given that the external field $\vec{H} = H\hat{z}$, and using the transformations shown in appendix Eq.(A2), we get

$$\dot{q} = -m^2[g' H h_\phi + h' H_\theta (H_\theta + h_\theta)] = -m H_\theta \dot{\theta} \quad (17)$$

where in the last step we used the evolution of axial angle θ from Eq.(2). Thus the total heat absorbed over a time period τ would be $\Delta Q = V \int^\tau dt(-m H_\theta \dot{\theta})$. Comparing with Eq.(11), it is clear that $\Delta s_r^{(1)} = -\Delta Q/T$. Thus one may interpret $\Delta s_r^{(1)}$ as the entropy production due to heat absorbed by the reservoir. We recover heat as the quantity associated with time-reversed trajectories where the direction of external field $\vec{H}$ is also reversed, along with $\vec{m}$. The situation is equivalent to the requirement of reversal of external flow direction in Ref. [42].

However, the total entropy production in the reservoir, consistent with Fokker-Planck equation, $\Delta s_r = \Delta s_r^{(1)} + \Delta s_r^{(2)}$ where $\Delta s_r^{(2)} = (V/\eta\gamma T) \int^\tau dt H_\theta \sin\theta\, \dot{\phi}$ is due to the precessional motion, where the term $H_\theta \sin\theta\, \dot{\phi}$ has the dimension of torque times angular velocity. Note that the precessional part of entropy production is because of the unidirectional torque due to the external magnetic field $\vec{H}$, which breaks the time-reversal symmetry. While the resultant precessional current would be observable from possible microscopic measurement of trajectories, however, would not reflect in calorimetric measurements that gives ΔQ. Thus this could be interpreted as a *non-thermal* entropy production in reservoir.

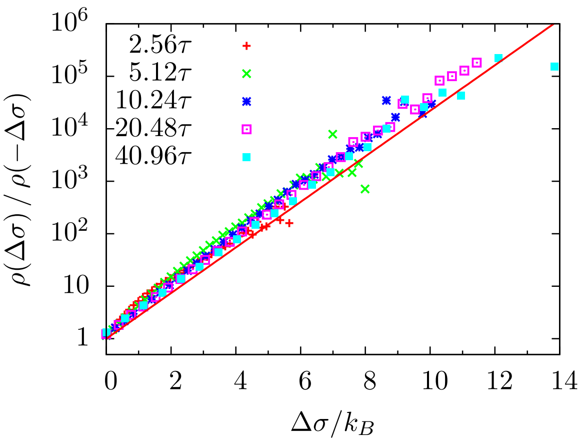

FIG. 4: (Color online) Ratio of probability distributions of positive and negative entropy production $\rho(\Delta s_t = \Delta\sigma)/\rho(\Delta s_t = -\Delta\sigma)$ calculated from the data described in the legend of Fig. 3. The solid line is a plot of the function $\exp(\Delta\sigma/k_B)$.

III. DISTRIBUTION OF ENTROPY PRODUCTION

In this section, we numerically evaluate the distribution of total entropy production $\Delta s_t = \Delta s + \Delta s_r$ over trajectories of various duration τ, at a steady state described by a constant external field H acting along z-direction. In this case $H_\theta = -H \sin\theta$ and $H_\phi = 0$. We use second order stochastic Runge-Kutta method to solve the LLG equations Eq.2. We perform the simulations by setting units such that the magnetization $m = 1$, strength of the magnetic field $H = 1$, temperature $k_B T = 1$ and using a time-step $\delta t = 0.01\tau$ such that $\tau = 1/\gamma$. To test the validity of the numerical integration we obtain the *equilibrium* distribution $W(\theta)$ that shows exact match with analytical result $W_0 \exp(-U(\theta)/k_B T)$ with $U(\theta) = -\cos\theta$ and $k_B T = 1$ (Fig.2(a)). A typical equilibrium trajectory is shown in Fig.1. This steady state supports a probability current $J_\phi(\theta)$ in the azimuthal direction [3] reflecting an overall precessional motion of the magnetization in clockwise direction around $\vec{H}$ (Fig.2(b)).

In Fig. 3 we show the probability distributions of entropy production $\rho(\Delta s_t)$ calculated from numerical integration of Eq. (2), and using Eq.s (14) and (15) for the expression of $\Delta s_t = \Delta s + \Delta s_r$. The distributions are calculated after collecting data over $10^7\tau_0$ for various durations of τ_0. Appreciable probability of negative entropy production is clearly visible. With increase in τ_0, the distribution broadens and the peak position shifts towards higher values of entropy. From each $\rho(\Delta s_t)$ curve, we obtain the ratio of probabilities $\rho(\Delta\sigma)/\rho(-\Delta\sigma)$ with $\rho(\Delta\sigma) = \rho(\Delta s_t = \Delta\sigma)$ and $\rho(-\Delta\sigma) = \rho(\Delta s_t = -\Delta\sigma)$. As is shown in Fig. 4, this ratio shows good agreement with the detailed fluctuation theorem $\rho(\Delta\sigma)/\rho(-\Delta\sigma) = \exp(\Delta\sigma/k_B)$.

IV. SUMMARY AND OUTLOOK

We obtained analytic expression for entropy production in a macrospin starting from the Landau-Lifshitz-Gilbert equations describing the system's stochastic dynamics in contact with a Langevin heat bath. Using the definition of time-dependent stochastic entropy, and the appropriate Fokker-Planck equation we showed that the total entropy production, after averaging, comes out to be positive definite, in accordance with the second law of thermodynamics. This calculation gave rise to a definition of entropy production in the reservoir, which we re-derived from time-reversed trajectory of the orientational dynamics of macrospin by treating the reversed protocol of external magnetic field as an even dynamical variable. This allowed us to derive an integral and detailed fluctuation theorem. We further presented, several other possibilities of conjugate trajectories and the resultant expressions of relative action that contribute towards total reservoir entropy. All these choices of conjugate trajectories lead to respective fluctuation theorems.

We then focused particularly on the simplest macrospin dynamics under an uniaxial time-independent external magnetic field $\vec{H} = H\hat{z}$. The system shows a curious dichotomy in its steady state behavior. While the distribution of polar angle $W(\theta)$ obeys the equilibrium Boltzmann distribution, the magnetization undergoes precessional motion along ϕ- direction leading to an azimuthal current $J_\phi(\theta)$ [3]. As a result the system produces entropy, even when the external filed $\vec{H}$ is time- independent. This is an interesting scenario, where one finds both an *equilibrium distribution* and *entropy production* in a single system. We verified these properties using direct numerical simulations. The distributions of entropy production, obtained from simulations, obey the detailed fluctuation theorem. While the contribution of *non-thermal* entropy production due to precessional motion may be measurable, e.g., by using Kerr microscopy [46, 47] to track the whole trajectories $[\theta(t), \phi(t)]$, it will not appear in calorimetric measurements as was shown by our energy balance condition. Whenever a system has circulation due to external torque, with displacements perpendicular to force such that the work done is zero, the circulation does not appear in energy balance, but would break time reversal symmetry [45, 48]. Thus the calculations we presented here regarding entropy production have wider implications. Our predictions for macrospin dynamics could be verified from experiments on super-paramagnetic particles.

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Appendix A: Ratio of path probabilities

Discretizing the trajectories into $i = 1, \ldots, N$ segments, the ratio of probabilities of time-forward and tim-reversed trajectories may be expressed as

$$\frac{\mathcal{P}_+}{\mathcal{P}_-} = \exp\left[-\frac{\delta t}{4D_0}\sum_i \{\vec{h}_i^2 - (\vec{h}_i^b)^2\}\right]. \tag{A1}$$

Here $\vec{h}_i$ denotes the stochastic field for time-forward trajectory, and $\vec{h}_i^b$ denotes the same for time-reversed path. In the above relation, it is understood that stochastic fields are replaced by dynamical variables using the equations of motion Eq. 2. The following relations between cartesian and spherical polar coordinates

$$\begin{aligned} h_x &= h_\theta \cos\theta\cos\phi - h_\phi \csc\theta\sin\phi \\ h_y &= h_\theta \cos\theta\sin\phi + h_\phi \csc\theta\cos\phi \\ h_z &= -h_\theta \sin\theta \end{aligned} \tag{A2}$$

lead to $\{\vec{h}_i^2 - (\vec{h}_i^b)^2\} = (h_\theta^2 - {h_\theta^b}^2) + (h_\phi^2 - {h_\phi^b}^2)\csc^2\theta$ evaluated at i-th instant. In writing the above coordinate transformations we ignored the radial component of stochastic magnetic field h_r, as it does not affect the angular dynamics of magnetization. The relation between time forward components of noise and dynamical variables are obtained from Eq.2

$$\begin{aligned} h_\theta &= \frac{c_1}{m}\dot\theta + \frac{c_2}{m}\sin\theta\,\dot\phi - H_\theta \\ h_\phi &= -\frac{c_2}{m}\sin\theta\,\dot\theta + \frac{c_1}{m}\sin^2\theta\,\dot\phi - H_\phi\sin\theta. \end{aligned} \tag{A3}$$

The relation obeyed by the corresponding time-reversed path

$$\begin{aligned} h_\theta^b &= -\frac{c_1}{m}\dot\theta - \frac{c_2}{m}\sin\theta\,\dot\phi - H_\theta \\ h_\phi^b &= \frac{c_2}{m}\sin\theta\,\dot\theta - \frac{c_1}{m}\sin^2\theta\,\dot\phi - H_\phi\sin\theta. \end{aligned} \tag{A4}$$

This ultimately leads to

$$\vec{h}_i^2 - (\vec{h}_i^b)^2 = -4\frac{H_\theta}{m}(c_1\dot\theta + c_2\sin\theta\,\dot\phi) - 4\frac{H_\phi}{m}(c_1\sin\theta\dot\phi - c_2\dot\theta).$$

Thus we obtain the relation

$$\begin{aligned} \frac{\mathcal{P}_+}{\mathcal{P}_-} &= \exp\left[\frac{\delta t}{mD_0}\sum_i \left\{H_\theta(c_1\dot\theta + c_2\sin\theta\dot\phi) + H_\phi(c_1\sin\theta\dot\phi - c_2\dot\theta)\right\}\right] \\ &= \exp\left[\frac{1}{mD_0}\int_0^{\tau_0} dt \left\{H_\theta(c_1\dot\theta + c_2\sin\theta\,\dot\phi) + H_\phi(c_1\sin\theta\dot\phi - c_2\dot\theta)\right\}\right]. \end{aligned} \tag{A5}$$

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